Spin Matrix for the Scaled Periodic Ising Model*

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August 6, 2010

In this paper we consider the matrix representation for the spin operator in the continuum limit of the periodic Ising model. We fix the interval $[-L, L]$ as the target for the continuum limit in the x variable with periodic boundary conditions $\sigma(L) = \sigma(-L)$ for the spin operator. The scaling limit we are interested in is a limit in which the lattice spacing, δ_T, for the Ising model in the interval $[-L, L]$ tends to 0 as the temperature $T \uparrow T_c$ approaches the critical temperature. The relationship between the lattice spacing and the temperature is determined by taking the inverse, δ_T^{-1}, to be the correlation length at temperature T. The interval $[-L, L]$ will then have length $2L$ in units of correlation length throughout the limiting process. The results for the matrix elements of the Ising spin operator will have consequences for the correlations in the continuum limit for the cylinder $[-L, L] \times \boldsymbol{R}$ and also for the torus $[-L, L] \times [-M, M]$.

We will not attempt to control the convergence of the scaling limit in this paper. This is partly because the convergence issue is much simplified if one uses formulas for the matrix elements on a finite periodic lattice that were conjectured by Bugrij and Lisovyy (see [1], [3] and [2]). A proof of these formulas has appeared but is quite complicated [5] and [6]. We expect that the technique we use in this paper can be extended to deal with the finite periodic lattice to give an alternative proof of the Bugrij-Lisovyy conjecture. The results for the continuum limit that we address in this paper were already announced in [10] and we should mention that the principal technique we employ is a Green function construction that we learned from the paper of Lisovyy [9].

We begin by recounting some results from Grethe Hystad's thesis [7], that provide a representation for the continuum limit of the Ising model (under the hypothesis that the Bugrij-Lisovyy conjecture is correct). The framework for her thesis is a reworking of Bruria Kaufmann's 1948 paper on the periodic Ising model [4] We can avoid some extraneous detail if we limit our considerations at the start to a finite lattice $\Lambda_{\ell,m} = \{-\ell, \cdots, \ell\} \times \{-m, \cdots, m\}$ where ℓ and m are positive integers. A configuration of spins on the lattice is a map,

$$\sigma : \Lambda_{\ell,m} \to \{-1, 1\}.$$

In this paper we are exclusively interested in the periodic boundary conditions $\sigma(-\ell, j) = \sigma(\ell, j)$ for all j. We will be interested both in the boundary conditions for periodic behavior in the vertical direction, $\sigma(j, -m) = \sigma(j, m)$ for all j and also in the cylindrical limit $m \to \infty$. The reader might note that little appears to be gained in the periodic situation by having the lattice sites run from $-\ell$ to ℓ rather than from 0 to ℓ as is perhaps customary. However, there are some differences in our treatment of the model for odd and even numbers of horizontal lattice sites that make it simpler to confine our

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attention to the odd case. The energy of a periodic configuration for the Ising model is,

$$E_{\ell,m}(\sigma) = -\sum_{\langle i,j\rangle} J_{i,j}\sigma(i)\sigma(j)$$

where the sum is over nearest neighbors i and j in $\Lambda_{\ell,m}$ and in our considerations there are only two interaction strengths, $J_{i,j} = J_1 > 0$ when i and j are horizontal neighbors and $J_{i,j} = J_2 > 0$ when i and j are vertical neighbors. Of course, in the periodic case lattice points with the same second coordinate and first coordinates ℓ and $-\ell$ are nearest neighbors as are points with the same first coordinate and second coordinates m and $-m$. The partition function at temperature T is the sum of the Boltzmann weights associated to a configuration,

$$Z_{\ell,m} = \sum_{\sigma} \exp(-E_{\ell,m}(\sigma)/k_B T).$$

The Boltzmann constant, k_B, appears in this formula, for reasons of tradition, but nothing is lost for us replacing J_j by J_j/k_B and setting $k_B = 1$.

Kaufmann's basic result is a formula for this partition function as the trace of the $2m+1$ power of a transfer matrix, V, that has a characterization in the spin representation of the orthogonal group,

$$Z_{\ell,m} = \mathrm{Tr}(V^{2m+1}).$$

1 Transfer Matrix

Our first goal is to summarize the reformulation of Kaufmann's result for V that can be found in the dissertation [7]. The transfer matrix V acts on the tensor product,

$$\mathcal{H} = \bigotimes_{n=-\ell}^{\ell} \boldsymbol{C}_n^2,$$

where $\boldsymbol{C}_n^2$ is just a copy of $\boldsymbol{C}^2$. Suppose that X is a map on $\boldsymbol{C}^2$. Let X_n denote the linear transformation on $\mathcal{H}$ that acts as X on the n^{th} slot and the identity in the remaining slots. There is a representation of the Clifford algebra on $\mathcal{H}$ determined by the action of generators,

$$q_n = \left(\prod_{k=-\ell}^{n-1} X_k\right) Y_n,$$
$$p_n = \left(\prod_{k=-\ell}^{n-1} X_k\right) Z_n, \tag{1}$$

where,

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Then $\{q_k, p_k\}$ satisfy the usual generator relations for the Clifford algebra,

$$p_k p_l + p_l p_k = 2\delta_{kl}, q_k q_l + q_l q_k = 2\delta_{lk}, q_k p_l + p_l q_k = 0.$$

We introduce the vector space W of complex linear combinations of q_k and p_k with coordinates,

$$W \ni w = \frac{1}{\sqrt{2}} \sum_{k=-\ell}^{\ell} x_k(w) q_k + y_k(w) p_k,$$

and distinguished complex bilinear form,

$$(w, w') = \sum_{k=-\ell}^{\ell} x_k(w) x_k(w') + y_k(w) y_k(w').$$

The $\{q_k, p_k\}$ are then generators of an irreducible representation of the Clifford algebra $\mathrm{Cliff}(W)$[11]. The conjugation,

$$\overline{w} = \frac{1}{\sqrt{2}} \sum_{k=-\ell}^{\ell} \overline{x}_k q_k + \overline{y}_k p_k,$$

determines an Hermitian inner product on W,

$$\langle w, w' \rangle = (\overline{w}, w').$$

The vectors q_k, p_k are real with respect to this conjugation and since they are self-adjoint with respect to the standard inner product on the tensor product of copies of the Hermitian inner product space $\boldsymbol{C}^2$ this representation of $\mathrm{Cliff}(W)$ is a $*-$ representation.

An important role in Kaufmann's analysis is played by the parity operator,

$$U = \prod_k X_k = \prod_k i p_k q_k.$$

Evidently $U^2 = 1$ and for reasons that will be apparent shortly we write $\mathcal{H}_A$ for the $+1$ eigenspace of U and $\mathcal{H}_P$ for the -1 eigenspace of U. The transfer matrix V respects the direct sum decomposition $\mathcal{H} = \mathcal{H}_A \oplus \mathcal{H}_P$ and we write,

$$V = V_A \oplus V_P.$$

In order to characterize the maps V_A and V_P we introduce the finite Fourier transforms $\mathcal{F}_A$ and $\mathcal{F}_P$. Let,

$$\mathcal{I}_\ell = \{-\ell, -\ell+1, \ldots, \ell\},$$

and note that in the $\{q_k, p_k\}$ basis we can think of W as the finite sequence space,

$$W = \ell^2(\mathcal{I}_\ell, \boldsymbol{C}^2).$$

For $f \in \ell^2(\mathcal{I}_\ell, \boldsymbol{C}^2)$ and $z \in \boldsymbol{C}$ write,

$$\mathcal{F}f(z) = \frac{1}{\sqrt{2\ell+1}} \sum_{k=-\ell}^{\ell} f(k) z^k$$

Let $\Sigma_A = \{z \in \boldsymbol{C} | z^{2\ell+1} = -1\}$ and $\Sigma_P = \{z \in \boldsymbol{C} | z^{2\ell+1} = 1\}$. We refer to Σ_A and Σ_P as the anti-periodic and periodic spectrum on the unit circle. Then $\mathcal{F}_A f$ and $\mathcal{F}_P f$ are respectively defined as the restrictions of $\mathcal{F}f$ to the anti-periodic and periodic spectral points on the unit circle,

$$\begin{aligned} \mathcal{F}_A f(z) &= \mathcal{F}f(z) \text{ for } z \in \Sigma_A, \\ \mathcal{F}_P f(z) &= \mathcal{F}f(z) \text{ for } z \in \Sigma_P. \end{aligned} \tag{2}$$

It is easy to confirm the inversion formulas,

$$f(k) = \frac{1}{\sqrt{2\ell+1}} \sum_{z\in\Sigma_A} \mathcal{F}_A f(z) z^{-k} \,\text{for}\, k \in \mathcal{I}_\ell,$$
$$f(k) = \frac{1}{\sqrt{2\ell+1}} \sum_{z\in\Sigma_P} \mathcal{F}_P f(z) z^{-k} \,\text{for}\, k \in \mathcal{I}_\ell.$$

For $|z| = 1$ define a 2×2 matrix,

$$T_z(V) = e^{-\gamma(z)} Q_+(z) + e^{\gamma(z)} Q_-(z), \text{with}\, Q_\pm(z) = \frac{1}{2}\begin{pmatrix} 1 & \mp w(z) \\ \mp w(z) & 1 \end{pmatrix}$$

where $\gamma(z)$ and $w(z)$ are defined by,

$$\operatorname{ch}\gamma(z) = \quad c_2^* c_1 - s_2^* s_1 (z + z^{-1})/2$$
$$w(z) = i\frac{\mathcal{A}_1(z)\mathcal{A}_2(z)}{\mathcal{A}_1(z^{-1})\mathcal{A}_2(z^{-1})}.$$

Note that for brevity we write,

$$\operatorname{ch} x = \cosh x, \text{and}\, \operatorname{sh} x = \sinh x.$$

The constants $s_j \,\text{and}\, c_j$ and their "duals" $s_j^* \,\text{and}\, c_j^*$ are defined by,

$$s_j = \operatorname{sh}(2J_j/k_B T), c_j = \operatorname{ch}(2J_j/k_B T),$$
$$s_j^* = s_j^{-1}, c_j^* = c_j s_j^{-1},$$

and finally,

$$\mathcal{A}_j(z) = \sqrt{\alpha_j - z}$$

with,

$$\alpha_1 = (c_1^* - s_1^*)(c_2 + s_2), \alpha_2 = (c_1^* + s_1^*)(c_2 + s_2)$$

It will be simplest for us to confine our attention to what happens when the temperature, T, is strictly less than the critical temperature ($T < T_c$)[8]. In this case $\alpha_2 > \alpha_1 > 1$ and smooth square roots $S^1 \ni z \to \mathcal{A}_j(z)$ are uniquely determined by the normalization, $\mathcal{A}_j(1) > 0$.

Let $T_A(V)$ denote the operator on W whose action in the $\mathcal{F}_A$ representation is given by,

$$\mathcal{F}_A f(z) \to T_z(V)\mathcal{F}_A f(z),$$

with a completely analogous definition for $T_P(V)$. Let $Q_A^\pm$ denote the operator on W whose action in the $\mathcal{F}_A$ representation is given by,

$$\mathcal{F}_A f(z) \to Q_\pm(z)\mathcal{F}_A f(z),$$

with an analogous definition for $Q_P^\pm$.

Both $T_A(V)$ and $T_P(V)$ have positive real spectrum and neither has 1 as an eigenvalue (for finite ℓ). Let $W_A^+ = Q_A^+ W$ denote the span of the eigenvectors for $T_A(V)$ that have eigenvalues between 0 and 1. and let $W_A^- = Q_A^- W$ denote the span of the eigenvectors for $T_A(V)$ that have eigenvalues greater than

1. Let T_A^+ denote the restriction of $T_A(V)$ to the subspace W_A^+. Make precisely analogous definitions for $W_P^\pm$ and T_P^+.

In the dissertation [7] it is proved that $\mathcal{H}_A$ is unitarily equivalent to the even tensor algebra over W_A^+. That is,

$$\mathcal{H}_A \simeq \mathrm{Alt}_{\mathrm{even}}(W_A^+) = \boldsymbol{C} \oplus W_A^+ \wedge W_A^+ \oplus \cdots \oplus \Lambda^{2\ell W_A^+},$$

where $\Lambda^k W_A^+$ is the k fold alternating tensor product of W_A^+ with itself, and that in this representation,

$$V_A = \lambda_A (1 \oplus T_A^+ \otimes T_A^+ \oplus \cdots \oplus (T_A^+)^{\otimes 2\ell}), \tag{3}$$

where λ_A is the largest eigenvalue of V_A given by,

$$\lambda_A = \exp \frac{1}{2} \sum_{z \in \Sigma_A} \gamma(z).$$

In a similar fashion $\mathcal{H}_P$ is unitarily equivalent to the even tensor algebra over W_P^+,

$$\mathcal{H}_P \simeq \mathrm{Alt}_{\mathrm{even}}(W_P^+) = \boldsymbol{C} \oplus W_P^+ \wedge W_P^+ \oplus \cdots \oplus \Lambda^{2\ell} W_P^+, \tag{4}$$

with,

$$V_P = \lambda_P (1 \oplus T_P^+ \otimes T_P^+ \oplus \cdots \oplus (T_P^+)^{\otimes 2\ell}),$$

where λ_P is the largest eigenvalue of V_P given by,

$$\lambda_P = \exp \frac{1}{2} \sum_{z \in \Sigma_P} \gamma(z).$$

Remark 1 This representation of $T_z(V)$ differs from the representation in [7] by conjugation by,

$$\begin{pmatrix} z & 0 \\ 0 & 1 \end{pmatrix},$$

representing translation by 1 lattice unit in the q_k basis elements (with $q_{\ell+1}$ equal to$-q_{-\ell}$, or $q_{-\ell}$ depending on which transform, $\mathcal{F}_A$ or $\mathcal{F}_P$, is relevant).

This is convenient since the corresponding "conjugation" acting on the induced rotation for the spin operator, reduces that operator to the "difference" of translations acting in the periodic and anti-periodic sectors.

2 Spin operator

The spin operator at $(j, 0)$, which we write as σ_j acts on the Clifford generators,

$$\begin{aligned} \sigma_j q_k \sigma_j^{-1} &= -\operatorname{sgn}(k-j-1) q_k \\ \sigma_j p_k \sigma_j^{-1} &= -\operatorname{sgn}(k-j-1) p_k \end{aligned} \tag{5}$$

Again, compared to [7] this is conjugated by $\begin{pmatrix} z & 0 \\ 0 & 1 \end{pmatrix}$. The spin operators σ_j anti-commute with U and hence map $\mathcal{H}_A$ into $\mathcal{H}_P$ and vice-versa. Thus we write,

$$\sigma_j = \begin{pmatrix} 0 & \sigma_j^{AP} \\ \sigma_j^{PA} & 0 \end{pmatrix} \text{acting on}\, \mathcal{H}_A \oplus \mathcal{H}_P.$$

One way to make use of (6) is to note that $\mathcal{H} = \mathcal{H}_A \oplus \mathcal{H}_P$ is also unitarily equivalent to both, $\mathrm{Alt}(W_A^+)$ and $\mathrm{Alt}(W_P^-)$. Thus we can regard σ_j as a map,

$$\sigma_j : \mathrm{Alt}(W_A^+) \to \mathrm{Alt}(W_P^+), \tag{6}$$

each space carrying an irreducible $*-$representation of the Clifford algebra $\mathrm{Cliff}(W)$. The unitary map, U_A, that allows us to (projectively) identify $\mathcal{H} = (\boldsymbol{C}^2)^{\otimes 2\ell+1}$ with $\mathrm{Alt}(W_A^+)$ intertwines the action of the Clifford algebra on $\mathcal{H}$ determined by the generators q_k, p_k with the Fock representation F_A on $\mathrm{Alt}(W_A^+)$. That is, for $w = \sum x_k q_k + y_k p_k$ we have,

$$U_A w = F_A(w) U_A. \tag{7}$$

Recall that the Fock representation is given by,

$$F_A(w) = c(w_A^+) + a(w_A^-), \text{where}\, w_A^\pm = Q_A^\pm w,$$

and $c(\cdot)$ and $a(\cdot)$ are creation and annihilation operators [8]. Since $Q_A^\pm$ are self-adjoint with respect to the Hermitian inner product on W, F_A determines a $*-$representation of $\mathrm{Cliff}(W)$, and since W is finite dimensional all such irreducible representations are unitarily equivalent. Thus $U_A : \mathcal{H} \to \mathrm{Alt}(W_A^+)$ exists and is determined up to a multiple of absolute value 1 by (7).

The map σ_j is then characterized up to a scalar multiple by the relations (6) and we can recover σ_j^{PA} by restricting (6) to the even subspace, $\mathrm{Alt}_{\mathrm{even}}(W_A^+)$. This is important for us since we are interested in understanding the matrix elements of σ_j in a basis of eigenvectors for the transfer matrix V and these are simple only in the spaces, $\mathrm{Alt}_{\mathrm{even}}(W_{A,P}^+)$. In a similar fashion we can regard,

$$\sigma_j : \mathrm{Alt}(W_P^+) \to \mathrm{Alt}(W_A^+),$$

and understand σ_j^{AP} as the restriction of σ_j to the even subspace $\mathrm{Alt}_{\mathrm{even}}(W_P^+)$. The reader should keep in mind, however, that this map is not to be confused with (6) – in fact, since $\sigma_j^2 = 1$, this map can be identified with σ_j^{-1} for σ_j coming from (6).

A fairly dramatic simplification of the induced rotation for σ_j occurs for $j = \ell$. In this case, (6) implies that the induced rotation for σ_ℓ is the identity. Thus σ_ℓ is an intertwining map for the Fock representations F_A and F_P. That is,

$$\sigma_\ell F_A(w) = F_P(w)\sigma_{\ell.} \tag{8}$$

One can recover the spin operators σ_k by conjugating with the appropriate "space" translations acting on,

$$\mathrm{Alt}_{\mathrm{even}}(W_A^+) \oplus \mathrm{Alt}_{\mathrm{even}}(W_P^+).$$

Translation by 1 lattice unit is given by,

$$1 \oplus (z \otimes z) \oplus \cdots \oplus z^{\otimes 2\ell}$$

in both even tensor algebras. However, since $z \in \Sigma_A$ in the first and $z \in \Sigma_P$ in the second, the spectrum of space translations is different in each summand. This is reflected by the induced rotation being $2\ell + 1$anti-periodic in $W = \ell^2(\mathcal{I}_m, \boldsymbol{C}^2)$ in the first summand and $2\ell + 1$ periodic in the second summand. This difference produces the signs in (6).

3 Eigenvectors of the Transfer Matrix

The operators $T_A(V)$ and $T_P(V)$ acting on W are self-adjoint. We introduce an orthonormal basis of eigenvectors for these maps. Define,

$$a(z) = \sqrt{\frac{\mathcal{A}_1(z)\mathcal{A}_2(z)}{\mathcal{A}_1(z^{-1})\mathcal{A}_2(z^{-1})}},$$

which we normalize so that it is positive for $z = 1$ (and as usual for us, $T < T_c$ so the square roots are all holomorphic functions in a neighborhood of $z \in S^1$). Then for $z \in \Sigma_A$, T_A has eigenvalues $\exp(-\gamma(z))$ and $\exp(\gamma(z))$ with associated eigenvectors $e_A^+(z)$ and $e_A^-(z)$, given as functions in the $\mathcal{F}_A$ representation of W by,

$$\begin{aligned} u \to e_A^+(z,u) &= \frac{1}{\sqrt{2}} \begin{pmatrix} a(z) \\ ia(z)^{-1} \end{pmatrix} \delta(z,u), \text{for}\, z,u \in \Sigma_A, \\ u \to e_A^-(z,u) &= \frac{1}{\sqrt{2}} \begin{pmatrix} a(z) \\ -ia(z)^{-1} \end{pmatrix} \delta(z,u), \text{for}\, z,u \in \Sigma_A. \end{aligned} \tag{9}$$

Here,

$$\delta(z,u) = \begin{cases} 1 \,\text{if}\, u = z \\ 0 \,\text{if}\, u \neq z \end{cases}$$

Then $\{e_A^+(z), e_A^-(z)\}_{z\in\Sigma_A}$ is an orthonormal basis for W with respect the Hermitian inner product and,

$$(e_A^+(z), e_A^-(z')) = \delta(z,z'),$$

so $\{e_A^+(z)\}_{z\in\Sigma_A}$ and $\{e_A^-(z)\}_{z\in\Sigma_A}$ are dual basis for W_A^+ and W_A^- with respect to the complex linear pairing between these two subspaces. We make exactly the same definitions for $e_P^{\pm}(z)$ except that $z \in \Sigma_P$ in this case.

We are interested in the matrix elements for the spin operator $\sigma = \sigma_\ell$ that connect the eigenvectors for V (all wedge products of the vectors $e_A^+(z)$ and $e_P^+(z)$). So, for example, let

$$\boldsymbol{z} = (z_1, z_2, \ldots, z_k) \,\text{with}\, z_i \in \Sigma_A \,\text{and}\, z_i \neq z_j \,\text{for}\, i \neq j;$$

and write $\boldsymbol{z} \in \Sigma_A$ in such circumstances.

If k is *even*, the vector $e_A^+(\boldsymbol{z})$ defined by,

$$e_A^+(\boldsymbol{z}) = e_A^+(z_1) \wedge \cdots \wedge e_A^+(z_k),$$

is an eigenvector for the transfer matrix V with eigenvalue,

$$\exp\left(\frac{1}{2}\sum_{z\in\Sigma_A}\gamma(z) - \sum_{z\in\boldsymbol{z}}\gamma(z)\right).$$

We write $z \in \boldsymbol{z}$ iff $z = z_j$ for some $j = 1, 2, \ldots, k$, and we write $k = \#\boldsymbol{z}$. Exchanging the roles of P and A let,

$$\boldsymbol{z}' = (z_1', z_2', \ldots, z_k') \,\text{with}\, z_i' \in \Sigma_P \,\text{and}\, z_i' \neq z_j' \,\text{for}\, i \neq j.$$

Then if k is *even*,

$$e_P^+(\boldsymbol{z}') = e_P^+(z_1') \wedge \cdots \wedge e_P^+(z_k')$$

is an eigenvector for the transfer matrix V with eigenvalue,

$$\exp\left(\frac{1}{2}\sum_{z'\in\Sigma_P}\gamma(z)-\sum_{z'\in\boldsymbol{z}'}\gamma(z')\right).$$

4 Matrix of the spin operator

The matrix element,

$$\langle e_P^+(\boldsymbol{z}'),\sigma e_A^+(\boldsymbol{z})\rangle \text{ for } \boldsymbol{z}'\in\Sigma_P \text{ and } \boldsymbol{z}\in\Sigma_A,$$

makes sense if we think of σ as a map,

$$\sigma:\mathrm{Alt}(W_A^+)\to\mathrm{Alt}(W_P^+).$$

Of course, $e_P^+(\boldsymbol{z}')$ and $e_A^+(\boldsymbol{z})$ are eigenvectors for V only if both $\#\boldsymbol{z}'$ and $\#\boldsymbol{z}$ are even. However it is easier to explain the version of Wick's theorem that allows us to reduce the calculation of these matrix to a few basic types if we admit matrix elements for arbitrary wedge products.

Let 0_A and 0_P denote the unit vacuum vectors in $\mathrm{Alt}(W_A^+)$ and $\mathrm{Alt}(W_P^+)$ (the reader might note that these vectors are only defined up a multiplier of absolute value 1 by the abstract unitary equivalence of these spaces with the original tensor product – we will have more to say about this later). The induced rotation $T(\sigma)$ is the identity on W. We write,

$$T(\sigma)=\begin{pmatrix} A & B\\ C & D\end{pmatrix},$$

for the matrix of this map taking one from the $W_A^+\oplus W_A^-$ splitting of W to the $W_P^+\oplus W_P^-$ splitting. So, for example $D=Q_P^-|_{W_A^-}$, maps W_A^- into W_P^-. We are particularly interested in the case where D is invertible. This happens precisely when $\langle 0_P,\sigma 0_A\rangle\neq 0$. In this circumstance one has the following formulas for the simplest matrix elements of σ [7],

$$\frac{\langle e_P^+(z'),\sigma e_A^+(z)\rangle}{\langle 0_P,\sigma 0_A\rangle}=(e_P^-(z'),D^{-\tau}e_A^+(z))=D^{-\tau}_{z',z}$$

$$\frac{\langle e_P^+(z_1')\wedge e_P^+(z_2'),\sigma 0_A\rangle}{\langle 0_P,\sigma 0_A\rangle}=(e_P^-(z_1'),BD^{-1}e_P^-(z_2'))=BD^{-1}_{z_1',z_2'}$$

and

$$\frac{\langle 0_P,\sigma e_A^+(z_1)\wedge e_A^+(z_2)\rangle}{\langle 0_P,\sigma 0_A\rangle}=(e_A^+(z_1),D^{-1}Ce_A^+(z_2))=D^{-1}C_{z_1,z_2}$$

Here, $D^\tau:W_P^+\to W_A^+$ is the transpose of D with respect to the bilinear form $(\cdot,\cdot)$ and

$$D^{-\tau}:W_A^+\to W_P^+,$$

is the inverse $(D^\tau)^{-1}$.

The following result is an extension of a well known reduction formula for the matrix elements of an element in the spin representation of the orthogonal group [8].

Theorem 1 *Let $\boldsymbol{z} \in \Sigma_A$ and $\boldsymbol{z}' \in \Sigma_P$. Then, supposing $\langle 0_P, \sigma 0_A \rangle \neq 0$,*

$$\frac{\langle e_P^+(\boldsymbol{z}'), \sigma e_A^+(\boldsymbol{z}) \rangle}{\langle 0_P, \sigma 0_A \rangle} = \mathrm{Pf} \begin{pmatrix} R_{\boldsymbol{z}' \times \boldsymbol{z}'} & R_{\boldsymbol{z}' \times \boldsymbol{z}} \\ R_{\boldsymbol{z} \times \boldsymbol{z}'} & R_{\boldsymbol{z} \times \boldsymbol{z}} \end{pmatrix},$$

where Pf *is the Pfaffian and the skew symmetric matrix R has matrix elements,*

$$(R_{\boldsymbol{z}' \times \boldsymbol{z}'})_{i,j} = \frac{\langle e_P^+(z_i') \wedge e_P^+(z_j'), \sigma 0_A \rangle}{\langle 0_P, \sigma 0_A \rangle} = BD^{-1}_{z_i', z_j'}$$

$$(R_{\boldsymbol{z}' \times \boldsymbol{z}})_{i,j} = -\left(R_{\boldsymbol{z} \times \boldsymbol{z}'}\right)_{j,i} = \frac{\langle e_P^+(z_i'), \sigma e_A^+(z_j) \rangle}{\langle 0_P, \sigma 0_A \rangle} = D^{-\tau}_{z_i', z_j}$$

and

$$(R_{\boldsymbol{z} \times \boldsymbol{z}})_{i,j} = \frac{\langle 0_P, \sigma e_A^+(z_i) \wedge e_A^+(z_j) \rangle}{\langle 0_P, \sigma 0_A \rangle} = D^{-1} C_{z_i, z_j}$$

This result is true even if $\#\boldsymbol{z}'$ and $\#\boldsymbol{z}$ are not even, but the matrix elements for σ no longer connect eigenvectors for the transfer matrix in that case. It is evident that to find explicit formulas for the spin matrix elements by this method one needs to calculate D^{-1}. We turn to the principal result in this paper, a calculation of D^{-1} in the periodic scaling limit.

5 The periodic scaling limit

As mentioned above the limit we are interested in is a continuum limit for a lattice theory that "lives" on the real domain $[-L, L] \times \boldsymbol{R}$, with a lattice spacing that tends to 0 as the temperature tends to the critical temperature. Periodic boundary conditions on $[-L, L]$ are maintained for the spins throughout. For the cylindrical case ($M \to \infty$) the correlations can be understood as 0_A vacuum expectations of products of spin operators [7]. The doubly periodic case the spin variable lives on the torus $[-L, L] \times [-M, M]/\sim$(the equivalence identifies $L \sim -L$ and $M \sim -M$ in the first and second factors respectively) and correlations are expressed as traces of the same products. In both cases one obtains explicit formulas that depend only on the formulas for the matrix elements of the spin operators. We examine what happens to the transfer matrix and the spin operator in this continuum limit without worrying too much about convergence questions. The thesis [7] contains results for the convergence problem assuming the Bugrij-Lisovyy conjecture [3].

The horizontal and vertical correlation lengths are known for the infinite volume Ising model. They are the reciprocals of the horizontal and vertical masses defined by,

$$m_1(T) = 2K_2 - 2K_1^*,$$
$$m_2(T) = 2K_1 - 2K_2^*,$$

where $K_j = J_j/k_BT$, and K_j^* is defined by,

$$\mathrm{sh}(2K_j)\,\mathrm{sh}(2K_j^*) = 1.$$

The limit we are interested in has J_j/k_B fixed and $T \uparrow T_c$, where the critical temperature, T_c, is defined by,

$$\mathrm{sh}\left(\frac{2J_1}{k_BT_c}\right)\mathrm{sh}\left(\frac{2J_2}{k_BT_c}\right) = 1.$$

The critical temperature is defined by the self-dual condition, $K_1^* = K_2$, and one sees that both $m_1(T)$ and $m_2(T)$ tend to zero as $T \uparrow T_c$. We look at the model with horizontal lattice spacing $m_1(T)$ and vertical lattice spacing $m_2(T)$. More specifically fix $L > 0$ and let $\ell(T) = [Lm_1(T)^{-1}]$ where $[x]$ is the greatest integer less than x. Consider the finite Fourier transform on,

$$W = \ell^2(\mathcal{I}_{\ell(T)}, \boldsymbol{C}^2),$$

written out in terms of the scaled variable $x \in m_1(T)\mathcal{I}_{\ell(T)} \subset [-L, L]$,

$$\mathcal{F}f(k) = \frac{1}{\sqrt{2\ell(T)+1}} \sum_{x \in m_1(T)\mathcal{I}_{\ell(T)}} f(x/m_1(T)) \exp\left(-\frac{2\pi i k}{2\ell(T)+1}\frac{x}{m_1(T)}\right).$$

The map,

$$f \to F(x) = \frac{1}{\sqrt{m_1(T)}} f(x/m_1(T)),$$

is a unitary map from the ℓ^2 space on $\mathcal{I}_{\ell(T)}$ to the ℓ^2 sequence space on $m_1(T)\mathcal{I}_{\ell(T)}$, with points weighted by the mass $m_1(T)$. The naive limit of this Fourier transform as $T \uparrow T_c$ is,

$$\mathcal{F}F(k) = \frac{1}{\sqrt{2L}} \int_{-L}^{L} F(x) \exp\left(-\frac{i\pi k x}{L}\right) dx, \tag{10}$$

where $k \in \boldsymbol{Z}$ in the case of the periodic Fourier transform and $k \in \boldsymbol{Z} + 1/2$ for the anti-periodic Fourier transform. Note that we have changed the look of the finite Fourier transform (2) by the substitution,

$$z \leftarrow \exp\left(-\frac{2\pi i k}{2\ell+1}\right) \text{where}\, k \in \boldsymbol{Z}(\text{periodic}) \,\text{or}\, k \in \boldsymbol{Z} + 1/2(\text{anti}-\text{periodic}).$$

The parametrization by $-k$ rather than k is motivated by the desire to get the usual shape for the Fourier series coefficients (10). The scaled vertical coordinate is $y \in m_2(T)\boldsymbol{Z}$. To understand the scaling limit we examine the "infinitesimal" generator γQ of $T(V)$. We are interested in the limit,

$$\lim_{T \uparrow T_c} m_2(T)^{-1} \gamma\left(e^{-\frac{2\pi i k}{2\ell(T)+1}}\right) Q\left(e^{-\frac{2\pi i k}{2\ell(T)+1}}\right),$$

where $k \in \boldsymbol{Z}$ or $k \in \boldsymbol{Z} + 1/2$ for the periodic or anti-periodic sectors respectively. In [8] it is shown that,

$$\lim_{T \uparrow T_c} m_2(T)^{-1} \gamma\left(e^{-\frac{2\pi i k}{2\ell(T)+1}}\right) = \sqrt{1+p^2}, \text{where}\, p = \frac{\pi k}{L}.$$

Recall that,

$$Q(z) = -\begin{pmatrix} 0 & w(z) \\ \overline{w(z)} & 0 \end{pmatrix}$$

with,

$$w(z) = i\frac{\mathcal{A}_1(z)\mathcal{A}_2(z)}{\mathcal{A}_1(z^{-1})\mathcal{A}_2(z^{-1})}.$$

Since $\alpha_2 > 1$ in the limit $T \uparrow T_c$ we see that,

$$\lim_{T \uparrow T_c} \frac{\mathcal{A}_2\left(e^{-\frac{2\pi i k}{2\ell(T)+1}}\right)}{\mathcal{A}_2\left(e^{\frac{2\pi i k}{2\ell(T)+1}}\right)} = 1.$$

However, $\alpha_1 = e^{m_1(T)}$ tends to 1 as $T \uparrow T_c$ and we have the asymptotics,

$$\mathcal{A}_1\left(e^{-\frac{2\pi i k}{2\ell(T)+1}}\right) \sim \sqrt{m_1(T)}\sqrt{1+ip}, \text{where}\, p = \frac{\pi k}{L}.$$

Thus,

$$\lim_{T\uparrow T_c} \frac{\mathcal{A}_1\left(e^{-\frac{2\pi i k}{2\ell(T)+1}}\right)}{\mathcal{A}_1\left(e^{\frac{2\pi i k}{2\ell(T)+1}}\right)} = \frac{\sqrt{1+ip}}{\sqrt{1-ip}}, \text{for}\, p = \frac{\pi k}{L}.$$

The induced rotation for the transfer matrix thus scales to the relation between infinitesimal vertical translation and infinitesimal horizontal translation,

$$\frac{\partial}{\partial x_2} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} p + \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix} \frac{\partial}{\partial x_1} + \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}.$$

where we identify ip with $\partial/\partial x_1$ based on (10). Consider the differential equation,

$$\frac{\partial \psi}{\partial x_2} + \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \frac{\partial \psi}{\partial x_1} + \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \psi = 0. \tag{11}$$

After making the substitution,

$$\psi \leftarrow \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \psi,$$

this differential equation becomes,

$$\begin{pmatrix} 1 & -2\partial \\ -2\bar{\partial} & 1 \end{pmatrix} \psi = 0, \tag{12}$$

where,

$$\partial = \frac{1}{2}\left(\frac{\partial}{\partial x_1} - i\frac{\partial}{\partial x_2}\right), \text{and}\, \bar{\partial} = \frac{1}{2}\left(\frac{\partial}{\partial x_1} + i\frac{\partial}{\partial x_2}\right).$$

Equation (12) is a convenient form for the (Euclidean) Dirac equation in two dimensions. In fact, it is useful to generalize this just slightly. We introduce a mass $m > 0$ into the Dirac equation,

$$\begin{pmatrix} m & -2\partial \\ -2\bar{\partial} & m \end{pmatrix} \psi = 0 \tag{13}$$

This arises naturally if one introduces the scaling variable mx_1 instead of x_1. The spin correlations depend only on the product mL so L and m are not really independent parameters. However, in the planar case the short distance behavior of the scaled correlations can be studied by analyzing the $m \to 0$ behavior of solutions to (13) and we expect something similar is possible in this case.

In the scaling limit it is natural to parametrize periodic and anti-periodic spectrum,

$$\Sigma_P = \left\{\frac{\pi}{L} n : n \in \boldsymbol{Z}\right\}, \tag{14}$$

and

$$\Sigma_A = \left\{\frac{\pi}{L}\left(n + \frac{1}{2}\right) : n \in \boldsymbol{Z}\right\}. \tag{15}$$

Of course, this notation conflicts with the earlier definition of $\Sigma_{P,A}$ but for the remainder of the paper we work with the scaling limit so the possibility of confusion is small. The scaling limit of the space W_A^+ can be identified as the space of solutions, $\psi(x) = \psi(x_1, x_2)$, to (13) that are $2L$ *anti-periodic* in x_1, are defined in the upper half plane $\Im(x) = x_2 > 0$ with $L^2[-L, L]$ boundary values for $x_2 = 0$ and tend to 0 as $x_2 \to +\infty$. W_A^- is similar but the solutions are defined for $x_2 < 0$ and tend to zero as $x_2 \to -\infty$. The limits for $W_P^\pm$ are similar but are defined in terms of $2L$ *periodic* solutions for (13). We understand σ_L as an interwining map,

$$\sigma_L : \mathrm{Alt}(W_A^+) \to \mathrm{Alt}(W_P^+).$$

This needs some further elaboration since in the scaling limit the Fock representations, F_A and F_P are not unitarily equivalent. Nonetheless, for the purposes of finding matrix elements of the molified spin operator $V^\varepsilon \sigma_L V^{\varepsilon'}$ (for $\varepsilon, \varepsilon' > 0$) and formulas for the spin correlations it suffices to invert the D matrix element of the identity map from $W_A^+ \oplus W_A^-$ to $W_P^+ \oplus W_P^-$. This is the problem to which we now turn.

6 The spectral curve for the Dirac equation

A crucial ingredient in our analysis of the scaling limit is a spectral curve for the Dirac equation that allows one to analytically continue the eigenfunctions that span $W_{A,P}^+$ to the eigenfunctions that span $W_{A,P}^-$.

For $u \in \boldsymbol{C}$ the function

$$\boldsymbol{C} \ni x \to e(x,u) := e^{-\frac{m}{2}(\bar{x}u + xu^{-1})} \begin{pmatrix} 1 \\ -u \end{pmatrix}$$

is an exponential solution to the Dirac equation

$$\mathcal{D}e = \begin{pmatrix} m & -2\partial \\ -2\bar{\partial} & m \end{pmatrix} e = 0,$$

where $x = x_1 + ix_2$ and $\partial = \partial_x = \frac{1}{2}(\partial/\partial x_1 - i\partial/\partial x_2)$. It is convenient to make the substitution $u = -e^{-s}$ and to regard the complex strip $\Im s \in [-\pi, \pi]$ with the upper and lower edges identified, $s + 2\pi i \simeq s$, as the "spectral curve" for the Dirac equation,$\mathcal{D}e = 0$. We write,

$$\Sigma(\mathcal{D}) = \boldsymbol{C}/2\pi i\boldsymbol{Z},$$

for this periodic strip.

For this parametrization the exponential solutions,

$$E(x,s) = e(x, -e^{-s}) = e^{\frac{m}{2}(\bar{x}e^{-s} + xe^{s})} \begin{pmatrix} 1 \\ e^{-s} \end{pmatrix}$$

that are purely oscillatory in the x_1 variable arise for $\Im s = \pm\frac{i\pi}{2}$. This fact is, in part, the reason we make the choice $u = -e^{-s}$. Define

$$\mathcal{M}_\pm = \left\{ s \in \boldsymbol{C} : \Im s = \pm\frac{i\pi}{2} \right\}.$$

Fourier analysis allows us to synthesize the solutions to the Dirac equation of interest to us as linear combinations of exponential solutions, $E(x,s)$, for $s \in \mathcal{M}_\pm$. Write

$$s_\pm = s \pm \frac{i\pi}{2}$$

Then it is easy to check that,

$$E(x,s_\pm) = \exp(m\,\mathrm{ch}(s_\pm)x_1 + im\,\mathrm{sh}(s_\pm)x_2)\begin{pmatrix} 1 \\ e^{-s_\pm} \end{pmatrix} = \exp(\pm im\,\mathrm{sh}(s)x_1 \mp m\,\mathrm{ch}(s)x_2)\begin{pmatrix} 1 \\ \mp ie^{-s} \end{pmatrix}$$

For s real we see that $E(x,s_+)$ is exponentially small at infinity for $x_2 > 0$ and $E(x,s_-)$ is exponentially small at infinity for $x_2 < 0$. It is useful to record a related result for the norm of $E(x,s)$. Let $x = x_1 + ix_2$ and $s = u + iv$ denote the splitting of x, and s into real and imaginary parts. Then after a short calculation one finds,

$$||E(x,s)|| = \sqrt{1+e^{-2u}}\exp(mx_1\,\mathrm{ch}(u)\cos(v) - mx_2\,\mathrm{ch}(u)\sin(v)). \tag{16}$$

7 Periodic and anti-periodic Green functions

Next we calculate the Green function for the operator $\mathcal{D}$ acting on the smooth $\boldsymbol{C}^2$ valued functions on,

$$[-L,L]\times \boldsymbol{R},$$

with either periodic or antiperiodic boundary conditions on $[-L,L]$. Introduce the Fourier coefficients,

$$\hat{f}(p,\xi) = \frac{1}{4\pi L}\int_{-L}^{L} dx_1 \int_{-\infty}^{\infty} dx_2 f(x_1,x_2)e^{-ipx_1}e^{-i\xi x_2}.$$

The inversion formula for smooth periodic functions in x_1 which are small at ∞ in x_2 is,

$$f(x_1,x_2) = \sum_{p\in\Sigma_P}\int_{-\infty}^{\infty} d\xi \hat{f}(p,\xi)e^{ipx_1}e^{i\xi x_2}$$

The inversion formula for smooth anti-periodic functions in x_1 which are small at ∞ in x_2 is,

$$f(x_1,x_2) = \sum_{p\in\Sigma_A}\int_{-\infty}^{\infty} d\xi \hat{f}(p,\xi)e^{ipx_1}e^{i\xi x_2}.$$

The Green function for $\mathcal{D}$ with a domain that contains the smooth periodic functions of $x_1 \in [-L,L]$ we denote by G_P. The Green function for $\mathcal{D}$ with a domain that contains the smooth anti-periodic functions of $x_1 \in [-L,L]$ we denote by G_A. For $x = x_1 + ix_2$ one finds,

$$G_{P,A}(x) = \frac{1}{4\pi L}\sum_{p\in\Sigma_{P,A}}\int_{-\infty}^{\infty} d\xi \frac{1}{m^2+p^2+\xi^2}\begin{pmatrix} m & ip+\xi \\ ip-\xi & m \end{pmatrix} e^{ipx_1 + i\xi x_2}.$$

If $x_2 > 0$ the ξ integration can be "closed" in the upper half plane and if $x_2 < 0$ the ξ integration can be "closed" in the lower half plane. One finds,

$$G_{P,A}(x) = \frac{1}{4L}\sum_{p\in\Sigma_{P,A}}\frac{1}{\omega(p)}\begin{pmatrix} m & i(p+\omega(p)) \\ i(p-\omega(p)) & m \end{pmatrix} e^{ipx_1 - \omega(p)x_2}\ \text{for}\ x_2 > 0, \tag{17}$$

and

$$G_{P,A}(x) = \frac{1}{4L} \sum_{p \in \Sigma_{P,A}} \frac{1}{\omega(p)} \begin{pmatrix} m & i(p-\omega(p)) \\ i(p+\omega(p)) & m \end{pmatrix} e^{ipx_1 + \omega(p) x_2} \text{ for } x_2 < 0. \tag{18}$$

To simplify some calculations with these Green functions we introduce,

$$g_{P,A}(x) = G_{P,A}(x) \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}.$$

The functions $g_{P,A}(x-x')$ have the advantage that their columns are solutions to the homogeneous Dirac equation as a function of x and their rows are solutions to the homogeneous Dirac equation as functions of x' (for $G_{P,A}$ one would need to introduce homogeneous solutions to the transpose of the Dirac equation to describe the rows).

Make the substitution $p = m \,\mathrm{sh}\, s$ in equation (17) and multiply on the right by $\begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}$. One finds,

$$g_{P,A}(x) = \frac{1}{4L} \sum_{s \in \Sigma'_{P,A}} \frac{1}{\mathrm{ch}\, s} \begin{pmatrix} e^{s} & i \\ -i & e^{-s} \end{pmatrix} e^{im\,\mathrm{sh}(s) x_1 - m\,\mathrm{ch}(s) x_2} \text{ for } x_2 > 0, \tag{19}$$

where we've written,

$$s \in \Sigma'_{P,A} \text{ for } m \,\mathrm{sh}\, s \in \Sigma_{P,A}. \tag{20}$$

Make the substitution $p = -m \,\mathrm{sh}\, s$ in equation (18) and multiply on the right by $\begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}$. One finds

$$g_{P,A}(x) = \frac{1}{4L} \sum_{s \in \Sigma'_{P,A}} \frac{1}{\mathrm{ch}\, s} \begin{pmatrix} -e^{s} & i \\ -i & -e^{-s} \end{pmatrix} e^{-im\,\mathrm{sh}(s) x_1 + m\,\mathrm{ch}(s) x_2} \text{ for } x_2 < 0 \tag{21}$$

Using (19), (21) and the definition of $E(x, s_\pm)$ above, one can verify,

$$g_{P,A}(x-x') = \frac{1}{4L} \sum_{s \in \Sigma'_{P,A}} \frac{e^{s}}{\mathrm{ch}\, s} E(x, s_+) E(x', s_-)^T \text{ for } \Im(x-x') > 0, \tag{22}$$

and

$$g_{P,A}(x-x') = -\frac{1}{4L} \sum_{s \in \Sigma'_{P,A}} \frac{e^{s}}{\mathrm{ch}\, s} E(x, s_-) E(x', s_+)^T \text{ for } \Im(x-x') < 0, \tag{23}$$

where X^T is the transpose of X. These formulas will be useful when we turn to our principal concern– a formula for the Green function of the Dirac operator on the strip $[-L, L] \times \boldsymbol{R}$with periodic boundary conditions in the lower half plane ($x_2 < 0$) and anti-periodic boundary conditions in the upper half plane.

We introduce periodic and anti-periodic spectral transforms that are closely related to the representations (22) and (23). For $f(x)$ a smooth periodic (P) or smooth anti-periodic (A) function of $x \in [-L, L]$, define spectral transforms $\mathcal{S}_P^\pm$ and $\mathcal{S}_A^\pm$

$$\begin{aligned} \mathcal{S}^+_{A,P} f(s) &= \frac{1}{2L} \int_{-L}^{L} e^{s/2} E(x, s_-)^T f(x) dx \quad \text{for } s \in \Sigma'_{A,P}, \\ \mathcal{S}^-_{A,P} f(s) &= \frac{1}{2L} \int_{-L}^{L} e^{s/2} E(x, s_+)^T f(x) dx \quad \text{for } s \in \Sigma'_{A,P}. \end{aligned} \tag{24}$$

The Fourier inversion formula implies that,

$$f(x) = \sum_{s\in\Sigma'_P} \frac{e^{s/2}}{2\,\mathrm{ch}(s)} \left(\mathcal{S}_P^+ f(s) E(x,s_+) + \mathcal{S}_P^- f(s) E(x,s_-)\right)$$

$$f(x) = \sum_{s\in\Sigma'_A} \frac{e^{s/2}}{2\,\mathrm{ch}(s)} \left(\mathcal{S}_A^+ f(s) E(x,s_+) + \mathcal{S}_A^- f(s) E(x,s_-)\right) \tag{25}$$

Where $f(x)$ is $2L$ periodic in the first case, and $2L$ anti-periodic in the second case (the formulas are true in an appropriate L^2, ℓ^2 sense even if the function $f(x)$ is neither periodic or anti-periodic but the convergence of the inversion sum is especially good if say the function f is smooth and (anti-)periodic and the spectral transform is the corresponding (anti-)periodic transform). The formulas (25) respectively determine the splitting of f in the $W_P^+ \oplus W_P^-$ and $W_A^+ \oplus W_A^-$ representations of $W = L^2[-L,L]$. The scaling limit analogues of the projections $Q_{A,P}^{\pm}$ are,

$$Q_{A,P}^{\pm} f(x) = \sum_{s\in\Sigma'_{A.P}} \frac{e^{s/2}}{2\,\mathrm{ch}\, s} \mathcal{S}_{A,P}^{\pm} f(s) E(x,s_{\pm}).$$

Again, since in this part of the paper we are exclusively concerned with the continuum limit there should be little danger of a confusion with the earlier usage for $Q_{A,P}^{\pm}$.

The Plancherel theorem implies,

$$\frac{1}{2L}\int_{-L}^{L} |f(x)|^2 dx = \sum_{s\in\Sigma'_P} \frac{1}{2\,\mathrm{ch}(s)} \left(|\mathcal{S}_P^+ f(s)|^2 + |\mathcal{S}_P^- f(s)|^2\right)$$

$$\frac{1}{2L}\int_{-L}^{L} |f(x)|^2 dx = \sum_{s\in\Sigma'_A} \frac{1}{2\,\mathrm{ch}(s)} \left(|\mathcal{S}_A^+ f(s)|^2 + |\mathcal{S}_A^- f(s)|^2\right) \tag{26}$$

It is useful to introduce the components of the inverse spectral transform, $\mathcal{R}$, as maps,

$$\mathcal{R}_A^{\pm} f(x) = \sum_{s\in\Sigma'_A} \frac{e^{s/2}}{2\,\mathrm{ch}(s)} f(s) E(x,s_{\pm}),$$

$$\mathcal{R}_P^{\pm} f(x) = \sum_{s\in\Sigma'_P} \frac{e^{s/2}}{2\,\mathrm{ch}(s)} f(s) E(x,s_{\pm}), \tag{27}$$

where $f \in \ell^2(\Sigma'_A)$ or $f \in \ell^2(\Sigma'_P)$ respectively. The Plancherel theorem implies that the maps $\mathcal{R}_{A,P}^{\pm}$ are all isometries from the appropriate square summable sequence space (with weight $(2\,\mathrm{ch}\, s)^{-1}$) into $L^2[-L,L]$.

In preparation for the construction of a Green function for the Dirac operator that mixes periodic and anti-periodic boundary conditions we follow Lisovyy [9] and turn next to the solution of a factorization problem on the spectral curve.

8 Factorization of σ_P/σ_A on the spectral curve

The function,

$$\sigma_P(s) = (e^{mL\,\mathrm{ch}\, s} - e^{-mL\,\mathrm{ch}\, s})/2$$

has simple 0's at the points $s = t \pm i\pi/2 \in \mathcal{M}_{\pm}$ with

$$t \in \Sigma'_P.$$

The function,

$$\sigma_A(s) = (e^{mL \operatorname{ch} s} + e^{-mL \operatorname{ch} s})/2$$

has simple 0's at the points $s = t \pm i\pi/2 \in \mathcal{M}_{\pm}$ with

$$t \in \Sigma'_A.$$

Their reciprocals have simple poles that can be used to express sums over the points in the periodic and anti-periodic spectrum as contour integrals. Lisovyy's construction of the Green function of interest to us depends on the solution of a factorization problem that we describe next. Consider the functions,

$$f_0(s) = \frac{\sigma_P(s)}{\sigma_A(s)} \,\text{defined for}\, -\epsilon < \Im(s) < \epsilon,$$

and,

$$f_\pi(s) = -\frac{\sigma_P(s)}{\sigma_A(s)} \,\text{defined for}\, -\pi \leqslant \Im(s) < -\pi + \epsilon \,\text{or}\, \pi - \epsilon < \Im(s) \leqslant \pi$$

The precise value of $\epsilon > 0$ is not important but we do want to choose $\epsilon < \pi/2$ to stay away from the zeros of $\sigma_A(s)$ and $\sigma_P(s)$. The functions $\sigma_A(s)$ and $\sigma_P(s)$ are $2\pi i$ periodic in s so we may regard f_0 as a function on $\Sigma(\mathcal{D})$ defined in a neighborhood of $\Im(x) = 0$ and f_π as a function on $\Sigma(\mathcal{D})$ defined in a neighborhood of $\Im(x) = \pm\pi$.

The change of sign in the definition of f_π is important because of the particular technique we use to produce a holomorphic factorization of the pair (f_0, f_π) on the spectral curve. As defined $f_0(s)$ tends to 1 as $\Re(s) \to \pm\infty$ on the line $\Im(s) = 0$ and $f_\pi(s)$ tends to 1 as $\Re(s) \to \pm\infty$ on the line $\Im(s) = \pm\pi$. In fact, it is useful to note that,

$$\begin{aligned} f_0(s) &= 1 + O(e^{-mL \operatorname{ch} s}) \,\text{for}\, \Im(s) = 0 \,\text{and}\, s \to \pm\infty \\ f_\pi(s + i\pi) &= 1 + O(e^{-mL \operatorname{ch} s}) \,\text{for}\, \Im(s) = 0 \,\text{and}\, s \to \pm\infty \end{aligned} \tag{28}$$

As a consequence both $\log(f_0)$ and $\log(f_\pi)$ will tend to 0 as $\Re(s) \to \pm\infty$ and this will insure the convergence of the integrals we use to define the additive splitting of the logarithms. In fact, since neither f_0 or f_π has either a zero or a pole in its simply connected domain of definition they both possess holomorphic logarithms which we can normalize to be 0 at ∞ in s. We henceforth denote these choices by $\log f_0$ and $\log f_\pi$. It follows from (28) that,

$$\begin{aligned} \log f_0(s) &= O(e^{-mL \operatorname{ch} s}) \,\text{for}\, \Im(s) = 0 \,\text{and}\, s \to \pm\infty \\ \log f_\pi(s + i\pi) &= O(e^{-mL \operatorname{ch} s}) \,\text{for}\, \Im(s) = 0 \,\text{and}\, s \to \pm\infty \end{aligned} \tag{29}$$

We can use the Green function for the Cauchy–Riemann operator on the spectral curve, $\Sigma(\mathcal{D})$, to additively split this pair in the usual fashion. Define the level sets for the imaginary part of s,

$$Y_a = \{s | \Im(s) = a\}$$

positively oriented by the one form ds. For $0 < \Im(x) < \pi$ define,

$$F_+(z) = \frac{1}{2\pi i} \int_{Y_0} \frac{e^s}{e^s - e^z} \log f_0(s) ds - \frac{1}{2\pi i} \int_{Y_\pi} \frac{e^s}{e^s - e^z} \log f_\pi(s) ds. \tag{30}$$

For $-\pi < \Im(z) < 0$ define,

$$F_-(z) = \frac{1}{2\pi i}\int_{Y_{-\pi}} \frac{e^s}{e^s - e^z}\log f_\pi(s)ds - \frac{1}{2\pi i}\int_{Y_0} \frac{e^s}{e^s - e^z}\log f_0(s)ds. \tag{31}$$

Because all the integrals involved converge absolutely, the functions $F_\pm(z)$ are holomorphic in their domains of definition. In fact, since

$$\frac{1}{2\pi i}\frac{e^s}{e^s - e^z}$$

is the kernal of the Green function for the Cauchy–Riemann operator on the spectral curve one can see that $F_+(z)$ and $F_-(z)$ have boundary values such that,

$$F_+(z) + F_-(z) = \log f_0(z)\,\text{for}\,\Im(z) = 0$$

and

$$F_+(z) + F_-(z) = \log f_\pi(z)\,\text{for}\,\Im(z) = \pm\pi,$$

where Y_π is identified with $Y_{-\pi}$ in the second equality of boundary values.

Once one knows this, it is clear that both $F_+(z)$ and $F_-(z)$ extend holomorphically to the strips $-\epsilon < \Im(z) < \pi + \epsilon$ and $-\pi - \epsilon < \Im(z) < \epsilon$ respectively since the domain of analyticity for $\log f_0$ and $\log f_\pi$ naturally enlarges the domain of analyticity for $F_\pm$. It is convenient to regard the enlarged domains for $F_\pm$ as subsets of the periodic strip $\Sigma(\mathcal{D}) = \boldsymbol{C}/2\pi i\boldsymbol{Z}$. Thus both $F_+(z)$ and $F_-(z)$ are holomorphic in a neighborhood of both $\mathrm{Im}(z) = 0$, and $\Im(z) = \pm\pi$ in $\Sigma(\mathcal{D})$.

Rewrite the formulas (30) and (31) using s to parametrize Y_0 and $s \pm i\pi$ to parametrize $Y_{\pm\pi}$. One finds,

$$F_+(z) = \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{1}{\mathrm{sh}(z-s)}\log\frac{\sigma_P(s)}{\sigma_A(s)}ds, \text{for}\, 0 < \Im(z) < \pi \tag{32}$$

and

$$F_-(z) = -\frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{1}{\mathrm{sh}(z-s)}\log\frac{\sigma_P(s)}{\sigma_A(s)}ds, \text{for}\, -\pi < \Im(z) < 0. \tag{33}$$

Since the hyperbolic sine is $i\pi$ anti-periodic it follows from this representation that,

$$F_+(z + i\pi) - F_-(z) = 0, \text{for}\, -\pi < \Im(z) < 0. \tag{34}$$

Next we introduce the modifications needed to holomorphically factor σ_P/σ_A with no sign change on $Y_{\pm\pi}$. Define,

$$\begin{aligned} \lambda_P(s) &= e^{s/2}\exp(F_+(s)), \text{for}\, -\varepsilon < \Im(s) < \pi + \varepsilon. \\ \lambda_A(s) &= e^{s/2}\exp(-F_-(s)), \text{for}\, -\pi - \varepsilon < \Im(s) < \varepsilon \end{aligned} \tag{35}$$

We regard both λ_P and λ_A as holomorphic functions defined on open subsets of $\Sigma(\mathcal{D})$. Keep in mind, however, that because $e^{s/2}$ is $2\pi i$ anti-periodic the representation for $\lambda_P(s)$ becomes,

$$\lambda_P(s) = -e^{s/2}\exp(F_+(s)), \text{for}\, -\pi \leqslant \Im(s) < -\pi + \varepsilon,$$

with a similar modification in the formula for $\lambda_A(s)$ for $\pi - \varepsilon < \Im(s) \leqslant \pi$.

With this $2\pi i$ anti-periodic adjustment one finds that,

$$\frac{\lambda_P(s)}{\lambda_A(s)} = \frac{\sigma_P(s)}{\sigma_A(s)} \quad \text{for}\, s\, \text{in}\, a\, \text{neighborhood of}\, Y_0\, \text{and}\, Y_\pi = Y_{-\pi} (\text{in}\, \Sigma(\mathcal{D})),$$

or more conveniently for us,

$$\frac{\lambda_P(s)}{\sigma_P(s)} = \frac{\lambda_A(s)}{\sigma_A(s)} \text{ for } s \text{ near } Y_0 \text{ and } Y_\pi = Y_{-\pi} (\text{in } \Sigma(\mathcal{D})). \tag{36}$$

9 The Green function $\mathcal{G}_{P/A}(x, x')$

Following Lisovyy [9] we write down a Green function for the Dirac operator on $[-L, L] \times \boldsymbol{R}$ which has periodic boundary conditions on $[-L, L] \times \boldsymbol{R}_+$ and anti-periodic boundary conditions on $[-L, L] \times \boldsymbol{R}_-$, where $\boldsymbol{R}_+$ is the set of positive real numbers and $\boldsymbol{R}_-$ is the set of negative real numbers. Define,

$$Y_b^a = Y_{b-a} - Y_{b+a},$$

which is the "counterclockwise" oriented boundary of the strip $b - a \leqslant \Im(z) \leqslant b + a$, centered at Y_b and having width $2a$. Next we introduce partial definitions for the Green function $\mathcal{G}_{P/A}(x, x')$ of principal interest for us. For $\Im(x) > 0 > \Im(x')$ define,

$$\mathcal{G}_{P/A}(x, x') = g_A(x - x') - \frac{im}{32\pi^2} \int_{Y^a_{\pi/2}} ds \int_{Y^b_{-\pi/2}} dt E(x, s) \frac{\lambda_P(s)}{\sigma_P(s)} \frac{e^s - e^t}{e^s + e^t} \frac{\lambda_A(t)}{\sigma_A(t)} E(x', t)^T. \tag{37}$$

For $\Im(x) < 0 < \Im(x')$ define,

$$\mathcal{G}_{P/A}(x, x') = g_P(x - x') - \frac{im}{32\pi^2} \int_{Y^a_{-\pi/2}} ds \int_{Y^b_{\pi/2}} dt E(x, s) \frac{\lambda_A(s)}{\sigma_A(s)} \frac{e^s - e^t}{e^s + e^t} \frac{\lambda_P(t)}{\sigma_P(t)} E(x', t)^T. \tag{38}$$

In these formulas we suppose that,

$$0 < b < a < \pi/2.$$

In both integrals above the difference $\Im(s - t)$ is $\pi \pm (a - b)$ or $\pi \pm (a + b)$ or $-\pi \pm (a - b)$ or $-\pi \pm (a + b)$. In no case is $e^s + e^t = 0$ since this would require that s and t have imaginary parts that differ by some odd multiple of π. Also, the restrictions on a and b keep s and t away from the zeros of σ_P and σ_A. Thus the integrals do not have any local singularities. Next we check convergence for large values of s and t. It will suffice to illustrate the estimate for the integrand in the s integration in (37). Write $s = u + iv$, and $x = x_1 + ix_2$. Using the easily confirmed observation that $F_\pm(s)$ tends to 0 as $|s| \to \infty$ we find the asymptotics ($u \to \pm\infty$),

$$\left|\frac{\lambda_P(s)}{\sigma_P(s)} E(x, s)\right| = O\left(\frac{\sqrt{e^u + e^{-u}}}{e^{mL \operatorname{ch}(u)}} \exp(mx_1 \operatorname{ch}(u) \cos(v) - mx_2 \operatorname{ch}(u) \sin(v))\right).$$

As long as $x_2 \geqslant 0, \sin(v) \geqslant 0$, and $|x_1| < L$, it follows that, for any ε such that $0 < \varepsilon < m(L - x_1 \cos(v))$,

$$\left|\frac{\lambda_P(s)}{\sigma_P(s)} E(x, s)\right| = O\left(\exp(-\varepsilon \operatorname{ch}(u))\right).$$

Since $\sin(\pi/2 \pm a) \geqslant 0$ for $0 < a \leqslant \pi/2$, the s integral in (38) converges at ∞ provided only that $x_2 \geqslant 0$ and $|x_1| < L$ (this is true even for the limiting case $a = \pi/2$). In a similar fashion one can see that for $x' = x'_1 + ix'_2$ the t integral in (38) converges at ∞ provided $x'_2 \leqslant 0$ and $|x'_1| < L$ (again even for $a = \pi/2$).

This is useful since we understand (37) and (38) in two ways. First by taking limits $a \to \pi/2$ and $b \to \pi/2$, and then by collapsing the integrals to the residues at the zeros of σ_P and σ_A.

First we take the limit $a \to \pi/2$ of the right hand side of (37) with b fixed. The analyticity of the integrand in s and the asymptotics for large s justify the replacement $Y^a_{\pi/2} \to Y_0 - Y_\pi$ in this limit. We cannot do the same with the limit $b \to \pi/2$ since the points $s \in Y_\pi$ and $t = s - i\pi \in Y_0$ and $s \in Y_0$ and $t = s - i\pi \in Y_{-\pi}$ are zeros of $e^s + e^t$. In the limit $b \to \pi/2$it is possible to skirt round $t = s - i\pi$ with a half circle of radius ε. The limit $\varepsilon \to 0$ then falls into two pieces. The first is a principal value integral and the second is a half residue. The principal value contribution to the limiting value of the integral on the right hand side of (37) is,

$$-\lim_{\varepsilon\to 0} \frac{im}{32\pi^2} \int_{\Gamma(\varepsilon)} dsdt E(x,s) \frac{\lambda_P(s)}{\sigma_P(s)} \frac{e^s - e^t}{e^s + e^t} \frac{\lambda_A(t)}{\sigma_A(t)} E(x',t)^T,$$

where,

$$\Gamma(\varepsilon) = (Y_0 - Y_\pi) \times (Y_{-\pi} - Y_0) \backslash \{(s,t) : |t - s + i\pi| < \varepsilon\}$$

The half residue contribution is,

$$\frac{m}{16\pi} \int_{Y_0 - Y_\pi} dsE(x,s) \frac{\lambda_P(s)\lambda_A(s - i\pi)}{\sigma_P(s)\sigma_A(s - i\pi)} E(x', s - i\pi)^T.$$

Consulting (34) we see that,

$$\lambda_P(s)\lambda_A(s - i\pi) = -ie^s,$$

so that this residue contribution is,

$$-\frac{im}{16\pi} \int_{Y_0 - Y_\pi} dsE(x,s) \frac{e^s}{\sigma_P(s)\sigma_A(s - i\pi)} E(x', s - i\pi)^T.$$

This integral collapses to a sum over the residues at $s \in \Sigma'_P + i\pi/2$ and $s \in \Sigma'_A + i\pi/2$. One finds (recall that $s_\pm = s \pm i\pi/2$),

$$\frac{1}{8L} \sum_{s\in\Sigma'_P} \frac{e^s}{\operatorname{ch} s} E(x,s_+)E(x',s_-)^T - \frac{1}{8L} \sum_{s\in\Sigma'_A} \frac{e^s}{\operatorname{ch} s} E(x,s_+)E(x',s_-)^T,$$

or consulting (22),

$$\frac{1}{2} g_P(x - x') - \frac{1}{2} g_A(x - x').$$

Thus for $\Im(x) > 0 > \Im(x')$,

$$\mathcal{G}_{P/A}(x,x') = g_{P+A}(x - x') - \lim_{\varepsilon\to 0} \frac{im}{32\pi^2} \int_{\Gamma(\varepsilon)} dsdt E(x,s) \frac{\lambda_P(s)}{\sigma_P(s)} \frac{e^s - e^t}{e^s + e^t} \frac{\lambda_A(t)}{\sigma_A(t)} E(x',t)^T, \qquad (39)$$

where,

$$g_{P+A} = (g_P + g_A)/2.$$

A precisely analogous calculation shows that for $\Im(x) < 0 < \Im(x')$,

$$\mathcal{G}_{P/A}(x,x') = g_{P+A}(x - x') - \lim_{\varepsilon\to 0} \frac{im}{32\pi^2} \int_{\Gamma'(\varepsilon)} dsdt E(x,s) \frac{\lambda_A(s)}{\sigma_A(s)} \frac{e^s - e^t}{e^s + e^t} \frac{\lambda_P(t)}{\sigma_P(t)} E(x',t)^T,$$

where $\Gamma'(\varepsilon) = (Y_{-\pi} - Y_0) \times (Y_0 - Y_\pi)\backslash\{(s,t) : |t - s - i\pi| < \varepsilon\}$. We transform the last integral using (36); we replace $\lambda_P(t)/\sigma_P(t)$ by $\lambda_A(t)/\sigma_A(t)$ and $\lambda_A(s)/\sigma_A(s)$ by $\lambda_P(s)/\sigma_P(s)$ and then use $Y_\pi = Y_{-\pi}$ on $\Sigma(\mathcal{D})$, to replace $\Gamma'(\varepsilon)$ with $\Gamma(\varepsilon)$. One finds that for $\Im(x) < 0 < \Im(x')$,

$$\mathcal{G}_{P/A}(x,x') = g_{P+A}(x-x') - \lim_{\varepsilon\to 0} \frac{im}{32\pi^2} \int_{\Gamma(\varepsilon)} dsdt E(x,s) \frac{\lambda_P(s)}{\sigma_P(s)} \frac{e^s - e^t}{e^s + e^t} \frac{\lambda_A(t)}{\sigma_A(t)} E(x',t)^T. \tag{40}$$

Note that in the formulas (39) and (40) the integrals on the right hand sides now have the same shape. This has important consequences for a polarization that we will associate with the kernel $\mathcal{G}_{P/A}(x,x')$.

We turn to the other evaluation of $\mathcal{G}_{P/A}(x,x')$ that is important for us. We can evaluate the integrals in (37) by residues at the poles on $s \in Y_{\pi/2}$ and $t \in Y_{-\pi/2}$ not forgeting the poles in $(e^s+e^t)^{-1}$ that arise for $s = t + i\pi$ (this pole gives rise to $-g_A(x-x')$). Since we want to maintain $a > b$, the $t$integral residue calculation should be done first. For $\Im(x) > 0 > \Im(x')$ we find,

$$\mathcal{G}_{P/A}(x,x') = \frac{i}{8mL^2} \sum_{s\in\Sigma_P^+} \sum_{t\in\Sigma_A^-} E(x,s) \frac{\lambda_P(s)}{\operatorname{sh} s} \frac{e^s - e^t}{e^s + e^t} \frac{\lambda_A(t)}{\operatorname{sh} t} e^{-mL(\operatorname{ch} s + \operatorname{ch} t)} E(x',t)^T.$$

where,

$$\Sigma_{P,A}^{\pm} = \Sigma'_{P,A} \pm i\pi/2.$$

Recalling that $s_\pm = s \pm i\pi/2$, we can turn this into a sum over real arguments, which makes it easier to see the convergence of the sum. For $\Im(x) > 0 > \Im(x')$,

$$\mathcal{G}_{P/A}(x,x') = \frac{i}{8mL^2} \sum_{s\in\Sigma'_P} \sum_{t\in\Sigma'_A} E(x,s_+) \frac{\lambda_P(s_+)}{\operatorname{ch} s} \frac{e^s + e^t}{e^s - e^t} \frac{\lambda_A(t_-)}{\operatorname{ch} t} e^{imL(\operatorname{sh} t - \operatorname{sh} s)} E(x',t_-)^T \tag{41}$$

An analogous residue calculation for $\Im(x) < 0 < \Im(x')$ shows that,

$$\mathcal{G}_{P/A}(x,x') = \frac{i}{8mL^2} \sum_{s\in\Sigma_A^-} \sum_{t\in\Sigma_P^+} E(x,s) \frac{\lambda_P(s)}{\operatorname{sh} s} \frac{e^s - e^t}{e^s + e^t} \frac{\lambda_A(t)}{\operatorname{sh} t} e^{-mL(\operatorname{ch} s + \operatorname{ch} t)} E(x',t)^T$$

And again this transforms into a sum on real arguments. For $\Im(x) < 0 < \Im(x')$,

$$\mathcal{G}_{P/A}(x,x') = \frac{i}{8mL^2} \sum_{s\in\Sigma'_A} \sum_{t\in\Sigma'_P} E(x,s_-) \frac{\lambda_A(s_-)}{\operatorname{ch} s} \frac{e^s + e^t}{e^s - e^t} \frac{\lambda_P(t_+)}{\operatorname{ch} t} e^{imL(\operatorname{sh} s - \operatorname{sh} t)} E(x',t_+)^T \tag{42}$$

The reader can now check that the representations (39) and (41) for $\mathcal{G}_{P/A}(x,x')$ extend continuously from the domain $\Im(x) > 0 > \Im(x')$ to $\Im(x) > 0 \geqslant \Im(x')$ and the representations (40) and (42) for $\mathcal{G}_{P/A}(x,x')$ extend continously from the domain $\Im(x) < 0 < \Im(x')$ to the domain $\Im(x) < 0 \leqslant \Im(x')$. In each case, we further require that $|x_1| < L$ and $|x'_1| < L$ to guarentee that the simple estimates we gave above assure pointwise convergence in the integral representations (the sums don't have a problem).

Associated with the Green function $\mathcal{G}_{P/A}$ there is a splitting of $L^2[-L,L]$ which is important for us. We will first discuss this in a naive setting and then introduce the estimates needed to make it work mathematically. We hope this will make it easier for the reader to see the reason for the definitions (37) and (38). For $x, x' \in [-L,L]$ define,

$$\mathcal{P}_{P/A}^+ f(x) = \lim_{\varepsilon\downarrow 0} \int_{-L}^{L} \mathcal{G}_{P/A}(x+i\varepsilon,x') f(x') dx' \tag{43}$$

and

$$\mathcal{P}^-_{P/A} f(x) = -\lim_{\varepsilon\downarrow 0}\int_{-L}^{L} \mathcal{G}_{P/A}(x - i\varepsilon, x') f(x') dx' \tag{44}$$

Note that (41) and (42) suggest that,

$$\mathcal{P}^+_{P/A} f \in W^+_P, \text{and}\, \mathcal{P}^-_{P/A} f \in W^-_A. \tag{45}$$

On the other hand (39) and (40) suggest that for $\Im(x) = \Im(x') = 0$,

$$\mathcal{G}_{P/A}(x + i0, x') - \mathcal{G}_{P/A}(x - i0, x') = g_{P+A}(x - x' + i0) - g_{P+A}(x - x' - i0) = \delta(x - x'), \tag{46}$$

since,

$$g_P(x - x' + i0) - g_P(x - x' - i0) = g_P(x - x' + i0) - g_P(x - x' - i0) = \delta(x - x').$$

Thus we expect that,

$$\mathcal{P}^+_{P/A} f + \mathcal{P}^-_{P/A} f = f. \tag{47}$$

We will prove later that,

$$\mathcal{P}^\pm_{P/A}\mathcal{P}^\mp_{P/A} = 0.$$

Combined with (47) this implies that $\mathcal{P}^\pm_{P/A}$ is a projection. These projections will allow us to explicitly invert the map,

$$Q^-_P : W^-_A \to W^-_P. \tag{48}$$

Note that if,

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

is the matrix of the identity map on W from the splitting $W^+_A \oplus W^-_A$ to $W^+_P \oplus W^-_P$ then (48) is just the map D.

To see how to invert D suppose that $f \in W^-_A$ and $Q^-_P f = g \in W^-_P$. Then $f = g^+_P + g$ where, $g^+_P \in W^+_P$, and so it follows from (45), (47) and the fact that $\mathcal{P}^-_{P/A}$ is a projection that is 0 on W^+_P that,

$$f = \mathcal{P}^-_{P/A} f = \mathcal{P}^-_{P/A}(g^+_P + g) = \mathcal{P}^-_{P/A} g = D^{-1} g \tag{49}$$

Thus (49) gives an explicit formula for the inversion of (48).

There are a number of matters that we glossed over in these "pointwise" calculations. The first matter we take up is a proof that $\mathcal{P}^\pm_{P/A}$ is bounded in L^2. It is enough to illustrate this for $\mathcal{P}^+_{P/A}$ using the formula (41). Suppose that f is a smooth compactly supported function on $[-L, L]$. Define,

$$F(x) = \frac{i}{8mL^2}\int_{-L}^{L} dx' \sum_{s\in\Sigma'_P}\sum_{t\in\Sigma'_A} E(x, s_+)\frac{\lambda_P(s_+)}{\operatorname{ch} s}\frac{e^s + e^t}{e^s - e^t}\frac{\lambda_A(t_-)}{\operatorname{ch} t} E(x', t_-)^T f(x').$$

Because we are interested in L^2 bounds we can and do ignore the unitary factor $e^{imL(\operatorname{sh} t - \operatorname{sh} s)}$ in (41). Using the formula (24) for the spectral tranform and the formulas (35) for λ_P and λ_A we find,

$$F(x) = \frac{i}{4mL}\sum_{s\in\Sigma'_P}\sum_{t\in\Sigma'_A} E(x, s_+) e^{s/2}\frac{e^{F_+(s_+)}}{\operatorname{ch} s}\frac{e^s + e^t}{e^s - e^t}\frac{e^{-F_-(t_-)}}{\operatorname{ch} t}\mathcal{S}^+_P f(t).$$

Introduce,

$$g(s) = \frac{i}{2mL} \sum_{t \in \Sigma'_A} e^{F_+(s_+)} \frac{e^s + e^t}{e^s - e^t} \frac{e^{-F_-(t_-)}}{\operatorname{ch} t} \mathcal{S}_P^+ f(t). \tag{50}$$

Then the formula (27) implies that,

$$F(x) = \mathcal{R}_P^+ g(s).$$

Because $\mathcal{R}_P^+$ is an isometry it will suffice to obtain a suitable estimate for,

$$\sum_{s \in \Sigma'_P} \frac{|g(s)|^2}{2 \operatorname{ch} s}.$$

However, since $\frac{e^s+e^t}{e^s-e^t}$, and $e^{F_+(s_+)-F_-(t_-)}$ are uniformly bounded in s and t for $s \in \Sigma'_P$ and $t \in \Sigma'_A$, and $(\operatorname{ch} t)^{-1}$ is summable for $t \in \Sigma'_P$ we can use the Cauchy-Schwartz inequality in (50) to see that,

$$\sum_{s \in \Sigma'_P} \frac{|g(s)|^2}{2 \operatorname{ch} s} \leqslant C \sum_{s \in \Sigma'_P} \frac{1}{\operatorname{ch} s} \sum_{t \in \Sigma'_P} \frac{|\mathcal{S}_P^+ f(t)|^2}{2 \operatorname{ch} t}.$$

This finishes the proof that $\mathcal{P}_{P/A}^+$ is bounded in L^2 on the smooth functions of compact support in $[-L, L]$. It then extends by density and continuity to a continuous map on all of L^2.

Next we show that $\mathcal{P}_{P/A}^+ f \in W_P^+$. Truncating the s sum to a finite range in the contribution that the double sum in (41) makes to the calculation of (43) we see that the result is clearly in W_P^+. Since the s sum converges in L^2 we see that $\mathcal{P}_{P/A}^+ f \in W_P^+$. The same argument shows that $\mathcal{P}_{P/A}^- f \in W_A^-$.

One other unresolved issue we wish to consider is the "pointwise" cancellation that produced (46). We do not have estimates for the terms that cancelled at the endpoints $x = \pm L$ and $x' = \pm L$. However, the L^2 continuity of the maps $\mathcal{P}_{P/A}^\pm$ makes it natural to define them as L^2limits defined first for functions, f, of compact support on $[-L, L]$. If f is such a function then the contribution made by the terms g_{A+P} in (39) and (40) to $\mathcal{P}_{P/A}^+ f + \mathcal{P}_{P/A}^- f$ is,

$$\frac{1}{2} \left(Q_P^+ + Q_A^+ \right) f + \frac{1}{2} \left(Q_P^- + Q_A^- \right) f = f.$$

Applied to a function of compact support in $[-L, L]$ the issue of convergence for the x' integration for the residual kernels in (39) and (40) in the calculation of $\mathcal{P}_{P/A}^\pm f$does not encounter any difficulties at $x' = \pm L$. The resulting functions clearly cancel except possibly at $x = \pm L$. However, since the kernels define bounded operators on L^2 the cancelation is good in L^2 and it follows that,

$$\mathcal{P}_{P/A}^+ f + \mathcal{P}_{P/A}^- f = f \text{ in } L^2.$$

Finally we take up the proof that $\mathcal{P}_{P/A}^-$ maps W_P^+ to 0. Suppose that,

$$f(x) = \mathcal{R}_P^+ F(x),$$

where $F(s)$ is non zero for only finitely many $s \in \Sigma'_P$. Such f are dense in W_P^+ so it will suffice to prove that $\mathcal{P}_{P/A}^- f = 0$. Use (42) in the definition (44). One encounters the integral,

$$\int_{-L}^{L} E(x', t_+)^T f(x') dx'.$$

This can be rewritten as the integral of a one form,

$$\int_{-L}^{L} E_1(x', t_+) f_1(x') dx' + E_2(x', t_+) f_2(x') d\bar{x}'.$$

It is not hard to check that as a consequence of the fact that both $x' \to E(x', t_+)$ and $x' \to f(x')$ are well behaved solutions to the Dirac equation in the upper half plane, the integrand in this last integral is a closed one form. This means that the contour in the integral can be shifted from $\Im(x') = 0$ to $\Im(x') = R > 0$ for any R. However, both solutions tend to 0 as $R \to +\infty$ and it follows that the integral itself must vanish. This shows that $\mathcal{P}^-_{P/A} f = 0$ and finishes the proof. The only change needed in the proof that $\mathcal{P}^+_{P/A}$ maps W^-_A to 0 is the observation that the product of two anti-periodic functions is periodic. This allows the contour deformation argument to proceed.

Next we turn to an an extension of the definition of $\mathcal{G}_{P/A}(x, x')$ to the remaining possible values for the argument (x, x'). First we consider the extension of $\mathcal{G}_{P/A}(x, x')$ to $\Im(x) > \Im(x') \geqslant 0$. Start with (41). In that equation replace the t sum by a contour integral on the boundary of tubular neighborhood of $\Im(t) = -\pi/2$ using the kernel $\sigma_P(t)^{-1}$. Note that the contour integral that it is natural to introduce contains an "extra" pole at the zero of $e^s + e^t$. We compensate by adding $g_P(x - x')$ to the contour integral to obtain a precise representation of the original sum. Suppose that $\Im(x) > 0$ and $\Im(x') = 0$. Extend the contours in the t integral to the horizontal lines $\Im(t) = 0, -\pi$. Replace the ratio $\lambda_A(t)/\sigma_A(t)$ in the integrand by $\lambda_P(t)/\sigma_P(t)$ using equation (36). Then collapse the resulting integral to the sum of the residues on $\Im(t) = \pi/2$. Keeping in mind that the orientation of the contour flips going from $\Im(t) = -\pi/2$ to $\Im(t) = \pi/2$ one finds,

$$\mathcal{G}_{P/A}(x, x') = g_P(x - x') - \frac{i}{8mL^2} \sum_{s \in \Sigma^+_P} \sum_{t \in \Sigma^+_P} E(x, s) \frac{\lambda_P(s)}{\operatorname{sh} s} \frac{e^s - e^t}{e^s + e^t} \frac{\lambda_P(t)}{\operatorname{sh} t} e^{-mL(\operatorname{ch} s + \operatorname{ch} t)} E(x', t)^T. \quad (51)$$

In this form the second term on the right hand side has a continuous extension to $\Im(x') \geqslant 0$. A little thought shows that the right hand side of (51) represents the Green function of interest when both x and x' are in the upper half plane. An almost identical argument produces the following formula for $\mathcal{G}_{P/A}(x, x')$ when both x and x' are in the lower half plane,

$$\mathcal{G}_{P/A}(x, x') = g_A(x - x') - \frac{i}{8mL^2} \sum_{s \in \Sigma^-_A} \sum_{t \in \Sigma^-_A} E(x, s) \frac{\lambda_A(s)}{\operatorname{sh} s} \frac{e^s - e^t}{e^s + e^t} \frac{\lambda_A(t)}{\operatorname{sh} t} e^{-mL(\operatorname{ch} s + \operatorname{ch} t)} E(x', t)^T. \quad (52)$$

We finish this section by using the projections $\mathcal{P}^\pm_{P/A}$ to give formulas for BD^{-1}, and $D^{-1}C$. Observe first that since,

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

is the matrix of the identity, every element $f \in W^-_A$ can be written,

$$f = Df + Bf,$$

or since D is invertible,

$$W^-_A \ni f = g + BD^{-1} g, \text{where}\, g \in W^-_P \text{ and}\, BD^{-1} g \in W^+_P.$$

This is described by saying that W_A^- is the graph of $BD^{-1} : W_P^- \to W_P^+$ over W_P^-. Now apply $\mathcal{P}_{P/A}^+$ to both sides of this last equation and use the fact that W_A^- is in the kernel of this map and that the elements in W_P^+ are fixed by this map. One finds,

$$0 = \mathcal{P}_{P/A}^+ g + BD^{-1} g \,\text{for}\, g \in W_P^-.$$

Thus,

$$BD^{-1} g = -\mathcal{P}_{P/A}^+ g \,\text{for}\, g \in W_P^- \tag{53}$$

Note that (51) is a particularly effective representation for $\mathcal{G}_{P/A}$ in the evaluation of this representation for BD^{-1}. In particular the g_P term in (51) makes no contribution since $Q_P^+ g = 0$ for $g \in W_P^-$.

Next we consider a representation for $D^{-1}C$. Observe first that since the identity map is an orthogonal map we have,

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} D^\tau & B^\tau \\ C^\tau & A^\tau \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \tag{54}$$

so that,

$$\begin{pmatrix} D^\tau & B^\tau \\ C^\tau & A^\tau \end{pmatrix},$$

is the matrix of the identity from $W = W_P^+ \oplus W_P^-$ to $W = W_A^+ \oplus W_A^-$. Thus for $f \in W_P^+$ we have,

$$W_P^+ \ni f = D^\tau f + C^\tau f,$$

or introducing $g = D^\tau f \in W_A^+$ we have,

$$W_P^+ \ni f = g + C^\tau D^{-\tau} g \,\text{for}\, g \in W_A^+.$$

Using the identity $CD^\tau + DC^\tau = 0$ that follows from (54) this becomes the graph representation,

$$W_P^+ \ni f = g - D^{-1} C g \,\text{for}\, g \in W_A^+.$$

Applying $\mathcal{P}_{P/A}^-$ to both sides and noting that W_P^+ is in the null space and W_A^- is in the range of this projection we find,

$$D^{-1} C g = \mathcal{P}_{P/A}^- g \,\text{for}\, g \in W_A^+ \tag{55}$$

10 Matrix for $D^{-1}, BD^{-1}, \text{and}\, D^{-1}C$ in the spectral variables

In this section we translate the representations (49), (53) and (55) into the spectral variables (24) and (25). Then we make some adjustments in these representations that we held off because incorporating them earlier on would have forced us to keep track of branches for the square root of $\operatorname{sh} t$, or irritating factors of $\exp(i\pi/4)$ in the calculations of the preceeding sections.

We start with the calculations for D^{-1}. Substitute (42) in (49) and make obvious use of (24) and (25). One finds the matrix representation of D^{-1} is,

$$D^{-1} \sim e^{-mL \operatorname{ch} s} \frac{e^{-s/2} \lambda_A(s)}{2mL} \frac{e^s - e^t}{e^s + e^t} \frac{e^{-s/2} \lambda_P(t)}{\operatorname{sh} t} e^{-mL \operatorname{ch} t} \,\text{for}\, s \in \Sigma_A^-, \text{and}\, t \in \Sigma_P^+ \tag{56}$$

This should be understood in the following sense. Replace s by s_- with the new value of s chosen in Σ'_A. Replace t with t_+ with the new value of t chosen in Σ'_P. The matrix of D^{-1} in (56) is implicitly indexed by $\Sigma'_A \times \Sigma'_P$.

Substituting (51) in (53) one finds (using the convention just explained) that the matrix of BD^{-1} is,

$$BD^{-1} \sim -ie^{-mL\,\mathrm{ch}\,s}\frac{e^{-s/2}\lambda_P(s)}{2mL}\frac{e^s-e^t}{e^s+e^t}\frac{e^{-s/2}\lambda_P(t)}{\mathrm{sh}\,t}e^{-mL\,\mathrm{ch}\,t}\,\mathrm{for}\,s\in\Sigma_P^+,\,\mathrm{and}\,t\in\Sigma_P^+.$$

Substitute (52) in (55) and one finds for the matrix of $D^{-1}C$,

$$D^{-1}C \sim ie^{-mL\,\mathrm{ch}\,s}\frac{e^{-s/2}\lambda_A(s)}{2mL}\frac{e^s-e^t}{e^s+e^t}\frac{e^{-s/2}\lambda_A(t)}{\mathrm{sh}\,t}e^{-mL\,\mathrm{ch}\,t}\,\mathrm{for}\,s\in\Sigma_A^-,\,\mathrm{and}\,t\in\Sigma_A^-.$$

Next we get rid of the asymmetry in these results that comes from using the ℓ^2 space on the spectral side with weight $(2\,\mathrm{ch}\,t)^{-1}$. One can pass to the matrix representation in the unweighted ℓ^2 space by multiplying on the left by $(2\,\mathrm{ch}\,s)^{-\frac{1}{2}}$ and on the right by $(2\,\mathrm{ch}\,t)^{\frac{1}{2}}$ where $(s,t)\in\Sigma'_{A,P}\times\Sigma'_{A,P}$ in whichever combination is appropriate. Do this and then adjust the spectral transform further by multiplying by $e^{i\pi/4}$ in the $W_{A,P}^-$ subspaces and by $e^{-i\pi/4}$ in the $W_{A,P}^+$ subspaces. One finds,

Theorem 2 *Suppose that*

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

is the matrix of the identity map on W from the $W_A^+\oplus W_A^-$ splitting to the $W_P^+\oplus W_P^-$ splitting. Then relative to the modified *spectral–inverse spectral transforms,*

$$\begin{array}{ll} \mathcal{S}_{A,P}^+f(s) = \frac{e^{-i\pi/4}}{2L}\int_{-L}^{L}\frac{e^{s/2}}{\sqrt{2\,\mathrm{ch}\,s}}E(x,s_-)^Tf(x)dx & \mathrm{for}\,s\in\Sigma'_{A,P}, \\ \mathcal{S}_{A,P}^-f(s) = \frac{e^{i\pi/4}}{2L}\int_{-L}^{L}\frac{e^{s/2}}{\sqrt{2\,\mathrm{ch}\,s}}E(x,s_+)^Tf(x)dx & \mathrm{for}\,s\in\Sigma'_{A,P}. \end{array}$$

and

$$\begin{array}{l} f(x) = \sum_{s\in\Sigma'_P}\frac{e^{s/2}}{\sqrt{2\,\mathrm{ch}\,s}}\left(e^{i\pi/4}\mathcal{S}_P^+f(s)E(x,s_+)+e^{-i\pi/4}\mathcal{S}_P^-f(s)E(x,s_-)\right) \\ f(x) = \sum_{s\in\Sigma'_A}\frac{e^{s/2}}{\sqrt{2\,\mathrm{ch}\,s}}\left(e^{i\pi/4}\mathcal{S}_A^+f(s)E(x,s_+)+e^{-i\pi/4}\mathcal{S}_A^-f(s)E(x,s_-)\right) \end{array}$$

The matrix elements of D^{-1}, BD^{-1} and $D^{-1}C$ are given by,

$$D^{-1} \sim \frac{1}{2mL}e^{-mL\,\mathrm{ch}\,s_-}\frac{e^{-s_-/2}\lambda_A(s_-)}{\sqrt{\mathrm{ch}\,s}}\frac{e^{s_-}-e^{t_+}}{e^{s_-}+e^{t_+}}\frac{e^{-t_+/2}\lambda_P(t_+)}{\sqrt{\mathrm{ch}\,t}}e^{-mL\,\mathrm{ch}\,t_+}\,\mathrm{for}(s,t)\in\Sigma'_A\times\Sigma'_P,$$

$$BD^{-1} \sim -\frac{1}{2mL}e^{-mL\,\mathrm{ch}\,s_+}\frac{e^{-s_+/2}\lambda_P(s_+)}{\sqrt{\mathrm{ch}\,s}}\frac{e^{s_+}-e^{t_+}}{e^{s_+}+e^{t_+}}\frac{e^{-t_+/2}\lambda_P(t_+)}{\sqrt{\mathrm{ch}\,t}}e^{-mL\,\mathrm{ch}\,t_+}\,\mathrm{for}(s,t)\in\Sigma'_P\times\Sigma'_P,$$

$$D^{-1}C \sim -\frac{1}{2mL}e^{-mL\,\mathrm{ch}\,s_-}\frac{e^{-s_-/2}\lambda_A(s_-)}{\sqrt{\mathrm{ch}\,s}}\frac{e^{s_-}-e^{t_-}}{e^{s_-}+e^{t_-}}\frac{e^{-t_-/2}\lambda_A(t_-)}{\sqrt{\mathrm{ch}\,t}}e^{-mL\,\mathrm{ch}\,t_-}\,\mathrm{for}(s,t)\in\Sigma'_A\times\Sigma'_A.$$

We did not bother to introduce new notation for the modified transforms that appear in this theorem since these transforms (modified or not) will make no further appearance. We note that the spectral transforms introduced in theorem 2 are equivalent to the introduction of coordinates for the bases (in the *original* spectral transform),

$$e_{A,P}^{\pm}(s) = t\to\frac{e^{\pm i\pi/4}}{\sqrt{2\,\mathrm{ch}\,s}}\begin{pmatrix} e^{s/2} \\ \mp ie^{-s/2} \end{pmatrix}\delta(t,s)\,\mathrm{for}\,t,s\in\Sigma'_{A,P}.$$

We follow the developments that lead up to theorem 1. Suppose that $s'_j \in \Sigma_P$ for $j = 1, \ldots, m$. Define

$$e_P^+(\boldsymbol{s}') = e_P^+(s'_1) \wedge \cdots \wedge e_P^+(s'_m), \text{where}\, \boldsymbol{s} = (s'_1, \ldots, s'_m).$$

For $s_j \in \Sigma_A$ for $j = 1, \ldots, n$ define,

$$e_A^+(\boldsymbol{s}) = e_A^+(s_1) \wedge \cdots \wedge e_A^+(s_n).$$

We apply theorem 1 to the calculation of the multiparticle matrix elements for the spin operator.

$$\frac{\langle e_P^+(\boldsymbol{s}'), \sigma e_A^+(\boldsymbol{s})\rangle}{\langle 0_P, \sigma 0_A\rangle} = \mathrm{Pf}\begin{pmatrix} R_{\boldsymbol{s}'\times\boldsymbol{s}'} & R_{\boldsymbol{s}'\times\boldsymbol{s}} \\ R_{\boldsymbol{s}\times\boldsymbol{s}'} & R_{\boldsymbol{s}\times\boldsymbol{s}} \end{pmatrix}, \tag{57}$$

where Pf is the Pfaffian and the skew symmetric matrix R has matrix elements,

$$(R_{\boldsymbol{s}'\times\boldsymbol{s}'})_{i,j} = \frac{\langle e_P^+(s'_i) \wedge e_P^+(s'_j), \sigma 0_A\rangle}{\langle 0_P, \sigma 0_A\rangle} = BD^{-1}_{s'_i, s'_j}$$

$$(R_{\boldsymbol{s}'\times\boldsymbol{s}})_{i,j} = -\left(R_{\boldsymbol{s}\times\boldsymbol{s}'}\right)_{j,i} = \frac{\langle e_P^+(s'_i), \sigma e_A^+(s_j)\rangle}{\langle 0_P, \sigma 0_A\rangle} = D^{-\tau}_{s'_i, s_j}$$

and

$$(R_{\boldsymbol{s}\times\boldsymbol{s}})_{i,j} = \frac{\langle 0_P, \sigma e_A^+(s_i) \wedge e_A^+(s_j)\rangle}{\langle 0_P, \sigma 0_A\rangle} = D^{-1}C_{s_i, s_j}$$

Now theorm 1 does not apply because a unitary intertwining map, σ, does not exist as a map from the Fock representation on $\mathrm{Alt}(W_A^+)$ to the Fock representation on $\mathrm{Alt}(W_P^+)$. However, all the difficulty with the map can be handled by composing it with positive powers of the transfer matrix on both sides. The factor $\langle 0_P, \sigma 0_A\rangle$ remains undefined, however. There are more and less natural ways to fix this constant but the discussion makes more sense if one understands how to make a choice for the wave function renormalization in the control of the scaling limit. Since we do not want to take up this matter here we will set $\langle 0_P, \sigma 0_A\rangle = 1$ (which is one sensible way in which to scale the spin operator) and proceed with the evaluation of (57) using the formulas for the matrix elements in theorem 2. We begin by defining two $(m+n) \times (m+n)$ matrices. Let Λ denote the $(m+n) \times (m+n)$ diagonal matrix with non-zero entries,

$$\Lambda_{j,j} = \frac{e^{-mL\,\mathrm{ch}\, s'_{j+}}}{\sqrt{2mL\,\mathrm{ch}\, s'_j}} e^{-s'_{j+}/2} \lambda_P(s'_{j+})\, \text{for}\, j = 1, \ldots, m,$$

and

$$\Lambda_{j+m,j+m} = \frac{e^{-mL\,\mathrm{ch}\, s_{j-}}}{\sqrt{2mL\,\mathrm{ch}\, s_j}} e^{-s_{j-}/2} \lambda_A(s_{j-})\, \text{for}\, j = 1, \ldots, n.$$

Next introduce,

$$t_j = s'_{j-} = s'_j - i\pi/2, \text{for}\, j = 1, \ldots, m,$$

and

$$t_{j+m} = s_{j+} = s_j + i\pi/2, \text{for}\, j = 1, \ldots, n.$$

Define and $(m+n)\times(m+n)$ matrix, T, by,

$$T_{i,j} = \frac{e^{t_i} - e^{t_j}}{e^{t_i} + e^{t_j}} \,\text{for}\, i,j = 1,\ldots,m+n.$$

Then using the results of theorem 2 we see that,

$$\begin{pmatrix} R_{\boldsymbol{s}'\times\boldsymbol{s}'} & R_{\boldsymbol{s}'\times\boldsymbol{s}} \\ R_{\boldsymbol{s}\times\boldsymbol{s}'} & R_{\boldsymbol{s}\times\boldsymbol{s}} \end{pmatrix} = \Lambda T \Lambda = \Lambda T \Lambda^{\tau},$$

where the transpose Λ^{τ} of the diagonal matrix Λ is, of course equal to Λ. Thus we have,

$$\mathrm{Pf} \begin{pmatrix} R_{\boldsymbol{s}'\times\boldsymbol{s}'} & R_{\boldsymbol{s}'\times\boldsymbol{s}} \\ R_{\boldsymbol{s}\times\boldsymbol{s}'} & R_{\boldsymbol{s}\times\boldsymbol{s}} \end{pmatrix} = \mathrm{Pf}(\Lambda T \Lambda^{\tau}) = \det(\Lambda)\,\mathrm{Pf}(T).$$

See [8]. The determinant of Λ is just the product of the entries on the diagonal. To finish this calculation we observe that the Pfaffian of T has a product form,

$$\mathrm{Pf}(T) = \prod_{i<j}^{m+n} \frac{e^{t_i} - e^{t_j}}{e^{t_i} + e^{t_j}}. \tag{58}$$

This formula is doubtless well known but can be confirmed in the following manner. Let Z be an $N\times N$ matrix with entries,

$$Z_{i,j} = \frac{z_i - z_j}{z_i + z_j}.$$

Then Z is a skew-symmetric matrix and,

$$\mathrm{Pf}(Z) = \prod_{i<j}^{N} \frac{z_i - z_j}{z_i + z_j}. \tag{59}$$

Expand the left hand side by "minors" of the first row (there is a suitable expansion for Pfaffians [8]). Use this to determine the residues at the poles ($-z_k$ for $k \neq j$) of the left hand side in the variable z_1. We lose nothing by supposing to start that $z_i \neq z_j$ for $i \neq j$; this makes all the poles simple. These residues are Pfaffians that can be inductively evaluated using (59). Now compare these residues with the residues of the right hand side calculated directly. This inductive agreement needs only a check of the case $N = 2$ to be complete. Thus the right and left sides of (59) are rational functions with equal residues at their simple poles. This implies that the difference of the two sides is a polynomial in z_1. It is clear that both sides have finite limits as $z_1 \to \infty$ and so this polynomial must be a constant. Since both sides are 0 at $z_1 = z_2$ they must be the same for all z_1. Anytime $z_i = z_j$ for $i \neq j$, both sides of (59) are 0 so the equality is true quite generally.

We are now prepared to state the principal theorem of this paper.

Theorem 3 *Write $\boldsymbol{s}' = (s'_1,\ldots,s'_m)$ with $s'_j \in \Sigma'_P$ and $\boldsymbol{s} = (s_1,\ldots,s_n)$ with $s_j \in \Sigma'_A$. Recall that for $\#\boldsymbol{s}'{=}m$ and $\#\boldsymbol{s} = n$ both even,*

$$e_P^{+}(\boldsymbol{s}') = e_P^{+}(s'_1) \wedge \cdots \wedge e_P^{+}(s'_m),$$

and

$$e_A^{+}(\boldsymbol{s}) = e_P^{+}(s_1) \wedge \cdots \wedge e_P^{+}(s_n)$$

are both eigenvectors for the transfer matrix.

The matrix elements of the periodic scaling limit of the Ising spin operator $\sigma = \sigma_L$ *normalized so that* $\langle 0_P, \sigma 0_A\rangle = 1$ *and restricted to a map from* $\mathrm{Alt}_{\mathrm{even}}(W_A^+)$ *to* $\mathrm{Alt}_{\mathrm{even}}(W_P^+)$ *in the basis of eigenvectors* $e_P^+(\boldsymbol{s}')$ *and* $e_A^+(\boldsymbol{s})$ *for the transfer matrix is given by,*

$$\langle e_P^+(\boldsymbol{s}'), \sigma e_A^+(\boldsymbol{s})\rangle = \Lambda_P(\boldsymbol{s}') \prod_{i<j}^{m+n} \frac{e^{t_i} - e^{t_j}}{e^{t_i} + e^{t_j}} \Lambda_A(\boldsymbol{s})$$

where,

$$\Lambda_P(\boldsymbol{s}') = e^{-im\pi/4} \prod_{j=1}^{m} \frac{e^{-imL\,\mathrm{sh}\, s_j'}}{\sqrt{2mL\,\mathrm{ch}\, s_j'}} e^{-s_j'/2} \lambda_P(s_j' + i\pi/2),$$

$$\Lambda_A(\boldsymbol{s}) = e^{in\pi/4} \prod_{j=1}^{n} \frac{e^{imL\,\mathrm{sh}\, s_j}}{\sqrt{2mL\,\mathrm{ch}\, s_j}} e^{-s_j/2} \lambda_A(s_j - i\pi/2),$$

and

$$t_j = s'_{j-} = s'_j - i\pi/2, \mathrm{for}\, j = 1, \ldots, m,$$
$$t_{j+m} = s_{j+} = s_j + i\pi/2, \mathrm{for}\, j = 1, \ldots, n.$$

These matrix elements can be used to calculate correlation functions of the spin operators in the scaling limit. For the convergence of the resulting sums to be manifest, however, it is helpful if some non-zero power of the transfer matrix appears between any two spin operators. We will sketch the issues involved for the scaled two point function on the cylinder.

Return to the lattice formalism on a finite lattice. The two point function on a $(2\ell+1)\times(2m+1)$ doubly periodic lattice is the ratio of traces,

$$\langle \sigma_{p,q}\sigma_{0,0}\rangle = \frac{\mathrm{Tr}(\sigma_p V^q \sigma_0 V^n)}{\mathrm{Tr}(V^{2m+1})},$$

where $q+n = 2m+1$. Sending $m\to\infty$, fixing p and q one obtains the correlations in the cylindrical limit. In this limit the trace in the denominator is asymptotic to λ_A^{2m+1} where λ_A is the largest eigenvalue of the transfer matrix. We briefly resurrect the notation from (3); don't confuse this with the notation $\lambda_A(s)$ for the solution to the factorization problem. In the limit $n\to\infty$ the product $\lambda_A^{-n}V^n$ tends to the orthogonal projection onto the eigenvector, 0_A, associated with the largest eigenvalue of V. Thus in the cylindrical limit the correlation tends to,

$$\langle \sigma_{p,q}\sigma_{0,0}\rangle = \lambda_A^{-q}\langle 0_A, \sigma_p V^q \sigma_0 0_A\rangle.$$

If we introduce a normalized transfer matrix $\hat{V}_P$ (in the periodic sector) so that $V_P = \lambda_P \hat{V}_P$ (and the eigenvalue for $\hat{V}_P$ associated with 0_P is 1) then the cylindrical correlation becomes,

$$\langle \sigma_{p,q}\sigma_{0,0}\rangle = (\lambda_P/\lambda_A)^q \langle 0_A, \sigma_p \hat{V}_P^q \sigma_0 0_A\rangle. \tag{60}$$

On the finite lattice the spin operator, σ_0, is unitary and $\sigma_0^2 = 1$. It follows that as long as we work in orthonormal bases for $\mathrm{Alt}_{\mathrm{even}}(W_A^+)$ and $\mathrm{Alt}_{\mathrm{even}}(W_P^+)$ the matrix of σ_0 thought of as a map from $\mathrm{Alt}_{\mathrm{even}}(W_P^+)$ to $\mathrm{Alt}_{\mathrm{even}}(W_A^+)$ is the hermitian conjugate of the matrix of σ_0 thought of as a map from

$\mathrm{Alt}_{\mathrm{even}}(W_A^+)$ to $\mathrm{Alt}_{\mathrm{even}}(W_P^+)$. Replacing σ_p by $\hat{\sigma}_p = \sigma_p/\langle 0_A, \sigma_p 0_P\rangle$ and σ_0 by $\hat{\sigma}_0 = \sigma_0/\langle 0_P, \sigma_0 0_A\rangle$ in preparation for scaling, it is straightforward to understand that, apart from space translations, the matrices for $\hat{\sigma}_p$ and $\hat{\sigma}_0$ are hermitian conjugates (observe that, modulo the convergence of the scaling limit, this shows that it is enough to understand the matrix of the spin operator from $\mathrm{Alt}_{\mathrm{even}}(W_A^+)$ to $\mathrm{Alt}_{\mathrm{even}}(W_P^+)$ which is what we've focused on in this paper.) We can now use (60) to informally understand what happens in the scaling limit to $\langle \hat{\sigma}_{p,q}\hat{\sigma}_{0,0}\rangle$. Make the substitutions $q \leftarrow qm_2^{-1}$ and $p \leftarrow pm_1^{-1}$. Note that in the limit we are interested in,

$$(\lambda_P/\lambda_A)^{qm_2^{-1}} \to e^{-\frac{\pi q}{4L}}.$$

Thus at least informally,

$$\langle \hat{\sigma}_{pm_1^{-1}, qm_2^{-1}}\hat{\sigma}_{0,0}\rangle \to e^{-\frac{\pi q}{4L}} \sum_{\boldsymbol{s}\in\Sigma'_P} z(\boldsymbol{s})^p \lambda(\boldsymbol{s})^q |\langle e_P^+(\boldsymbol{s}), \hat{\sigma} 0_A\rangle|^2,$$

where the sum is over all $\boldsymbol{s} = (s_1, s_2, \ldots, s_n)$ with $s_j \in \Sigma'_P$ and $s_i \neq s_j$ for $i \neq j$ (including $\boldsymbol{s} = \emptyset$). Space translation and vertical transfer are given by,

$$z(\boldsymbol{s}) = \prod_{j=1}^{n} \exp(im\,\mathrm{sh}(s_j)),$$

$$\lambda(\boldsymbol{s}) = \prod_{j=1}^{n} \exp(-m\,\mathrm{ch}(s_j)).$$

The phase factors $z(\boldsymbol{s})^p$ are the only surviving ones in the calculation and one can write,

$$|\langle e_P^+(\boldsymbol{s}), \hat{\sigma} 0_A\rangle|^2 = \left| \prod_{j=1}^{n} \frac{e^{-s_j/2}\lambda_P(s_j + i\pi/2)}{\sqrt{2mL\,\mathrm{ch}\, s_j}} \prod_{i<j}^{n} \frac{e^{t_i} - e^{t_j}}{e^{t_i} + e^{t_j}} \right|^2$$

Results of this sort were announced in [10] for the scaling functions and derived in [1] for correlations on the finite periodic lattice.

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